Regarding the approximate solution of a system of nonlinear differential equations with a small parameter

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Abstract. The main problem in calculations and calculations of the trajectory of motion of bodies is the need to introduce such universal variables in which the right side of the equations of motion remains a holomorphic function of coordinates even at the moment of collision. When calculating perturbations, the method of small parameters is used, which made it possible to obtain approximate solutions in the vicinity of the starting point. When using regularizing variables, the accuracy of calculations increases.

1 Introduction

The main problem associated with measuring the trajectories of a body is the approximation of the rectangular coordinates of a given body using algebraic polynomials with respect to a regularizing variable. The main principles of deriving the perturbation theory in rectangular coordinates are given in scientific papers [1-2]. Different variants of universal variables that make it possible to make a transition without singularities from one conical section to another were considered in the article [2], ways of using universal variables in a number of problems of mechanics to determine perturbations by the method of variation of arbitrary constants are outlined. In [3], auxiliary functions in the form of power series with respect to a universal variable are introduced, as well as a method for sequentially obtaining the terms of these series is indicated and an error estimate is given when replacing infinite series with a finite number of their terms.

In articles [4-5], the issues of using universal variables in calculating passive flight trajectories are considered, the components of the initial values of the radius vector and velocity are used as osculating variables.

2 Problem specification and decision

Let us consider the system of differential equations with a small parameter at derivatives in the vector form:

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$$\varepsilon \frac{dx}{dt} = f(x,y), \qquad \frac{dy}{dt} = g(x,y), \tag{1}$$

where $x = (x^1,...,x^k), y = (y^1,...,y^l)$.

Functions $f(x,y) = \left\{ f^1(x,y),..., f^k(x,y) \right\}$,
$g(x,y) = \left\{ g^1(x,y),..., g^l(x,y) \right\}$ are defined and
manyfold differentiable in a certain domain of variables (x,y), ε – small parameter, $\varepsilon > 0$.

The behavior of solutions of the system (1) when $\varepsilon \to 0$ essentially depends on the properties of stationary solutions of the system of fast motions:

$$\frac{dx}{d\tau} = f(x,y) \tag{2}$$

where $\tau = \frac{t}{\varepsilon}$, y - is a constant vector.

In order to study the dependance of stationary solutions of the system (2) on a vector parameter *y*, we will use a lemma below.

Lemma. Let us consider a system of equations:

$$\Phi^1(x,y) = ... = \Phi^k(x.y) = 0 \tag{3}$$

Let the following conditions (4) be met:

- functions $\Phi^i(x,y), i = 1,...,k$ are defined and manyfold differentiable in a certain domain containing point (x_0, y_0),
- point (x_0, y_0) satisfies the system (3), i. e.

$$\Phi^i(x_0, y_0) = 0, i = 1,...,k \tag{4}$$

- matrix $A = \left\| \frac{\partial \Phi^i(x_0,y_0)}{\partial x} \right\|$ has zero eigenvalue of multiplicity one,
- $p = n_\delta \Phi^\delta_{x^\alpha x^\beta}(x_0, y_0) m^\alpha m^\beta \neq 0$,

where $m = (m^1,...,m^k), n = (n_1,...,n_k) -$ are eigenvectors of matrix A and the transpose of matrix A' correspondingly, which satisfy zero eigenvalue and are normalized in such a way that $m^\alpha n_\alpha = 1$, and let us assume $m^1 \neq 0, n_1 \neq 0$.

If the conditions (4) are met, then:

1. System (5)

$$\Phi^2(x,y) = \ldots = \Phi^k(x,y) = det\left\|\frac{\partial \Phi^i}{\partial x^j}\right\| = 0 \tag{5}$$

explicitly determines variables $x^1,\ldots,x^k$ as functions $y^1,\ldots,y^l$ in the neighborhood of point y_0. When substituting these functions into $\Phi^1(x,y)$, we obtain function $\chi(y)$ with the following properties:

$$\chi(y_0) = 0, \frac{\partial \chi(y_0)}{\partial y^j} = n_\delta \Phi^\delta_{y^j}(x_0, y_0), j = 1,\ldots,l. \tag{6}$$

2. If at least one of the derivatives $\frac{\partial \chi(y_0)}{\partial y^j}$ is non-zero, then, in a certain neighborhood of point y_0, equation $\chi(y) = 0$ defines smooth surface π, which passes through this point. When y passes through surface π, function $\chi(y)$ changes its sign. In this case, when the values of y are quite close to y_0 and satisfy the condition $p\chi(y) \le 0$, two functions are defined as follows: $x = \psi_1(y), x = \psi_2(y)$. These functions satisfy the system (3), take equal vales in the points located on surface π and have continuous partial derivatives in the points, which do not belong to surface π.

Proof.

Let us assume $x_0 = 0, y_0 = 0, n_1 = 1$. Proceeding to new variables $\xi^1,\ldots,\xi^k$ in the system (3), $x = \xi^i \overline{e}_i$, where $\overline{e}_1 = m$, while the rest $\overline{e}_i, (i = 2,\ldots,k)$ satisfy condition $(\overline{e}_i, n) = 0$, we obtain the following system of equations:

$$F^1(\xi,y) = \ldots = F^k(\xi,y) = 0, \quad F^i(\xi,y) = f^i(\xi^j \cdot \overline{e}_j, y), (i,j = 1,\ldots,k) \tag{7}$$

The expansion of functions $F^i(\xi,y)$ by their Taylor series in the neighborhood of point $\xi = 0, y = 0$ is written as:

$$F^i(\xi,y) = d^i \cdot (\xi^1)^2 + c^i_{\alpha'} \cdot \xi^{\alpha'} + r^i_\beta \cdot y^\beta + \ldots, \tag{8}$$

where $d^i = \frac{1}{2} f^i_{x^\alpha x^\beta}(0,0) m^\alpha m^\beta, c^i_{\alpha'} = f^i_{x^\beta}(0,0) e^\beta_{\alpha'}$ (9)

$$r^i_\beta = f^i_{y^\beta}(0,0), \quad det\left\|c^{\beta'}_{\alpha'}\right\| \ne 0, (i,\alpha,\beta = 1,\ldots,k), (\alpha',\beta' = 2,\ldots,k) \tag{10}$$

Let us consider a system of equations:

$$F^2(\xi,y) = \ldots = F^k(\xi,y) = 0 \tag{11}$$

As long as $det\left\|\frac{\partial F^{\alpha'}(0,0)}{\partial \xi^{\beta'}}\right\| = det\left\|c_{\beta'}^{\alpha'}\right\| \neq 0,$ then in a certain neighborhood of point $\xi = 0, y = 0$ the system (11) uniquely determines functions

$$\xi^2 = \tilde{\xi}^2(\xi^1, y), \ldots, \xi^k = \tilde{\xi}^k(\xi^1, y). \tag{12}$$

These functions (12) satisfy condition $\tilde{\xi}^{\alpha'}(0,0) = 0, (\alpha' = 2, \ldots, k)$ and have continuous partial derivatives (13), (14).

$$c_{\alpha'}^{i} \cdot \frac{\partial \tilde{\xi}^{\alpha'}(\xi^1, y)}{\partial y^{\beta}} + f_{y^{\beta}}^{i}(0,0) = 0, (i = 2, \ldots, l), (\xi^1 = 0, y = 0) \tag{13}$$

$$\frac{\partial F^i}{\partial \xi^{\alpha'}} \cdot \frac{\partial \tilde{\xi}^{\alpha'}(\xi^1, y)}{\partial \xi^1} + \frac{\partial F^i}{\partial \xi^1} = 0, (i = 2, \ldots, l) \tag{14}$$

As long as $\frac{\partial F^i(0,0)}{\partial \xi^1} = 0,$ then

$$\frac{\partial \tilde{\xi}^{\alpha'}(0,0)}{\partial \xi^1} = 0, (i = 2, \ldots, l) \tag{15}$$

As a result of substitution of functions $\tilde{\xi}^{\alpha'}(\xi^1, y)$ into the first equation of the system (7), we obtain the following equation:

$$\psi(\xi^1, y) = 0 \tag{16}$$

where

$$\begin{gathered} \psi(\xi^1, y) = F^1(\xi^1, \tilde{\xi}^*(\xi^1, y), y) \\ \tilde{\xi}^*(\xi^1, y) = (\tilde{\xi}^2(\xi^1, y), \ldots, \tilde{\xi}^k(\xi^1, y)) \end{gathered} \tag{17}$$

Using the expansion (8), the conditions (15) and considering that $\psi(0,0) = 0$, we obtain:

$$\psi_{\xi^1}(\xi^1, y) = F_{\xi^1}^1 + F_{\xi^{\alpha'}}^1 \cdot \frac{\partial \xi^{\alpha'}}{\partial \xi^1}(\xi^1, y), (\alpha' = 2, \ldots, k) \tag{18}$$

Let us find the derivatives $\psi_{y^i}(0,0)$ using (17).

$$\psi_{y^i}(0,0) = F_{\xi^{\alpha'}}^1(0,0) \cdot \frac{\partial \tilde{\xi}^{\alpha'}(0,0)}{\partial y^i} + F_{y^i}^1(0,0) \tag{19}$$

If we substitute derivatives $\frac{\partial \tilde{\xi}^{\alpha'}(0,0)}{\partial y^i}$ determined by Cramer's rule from the system (13) into (19), derivative $\psi_{y^i}(0,0)$ can be represented as a determinant (20).

$$\psi_{y^i}(0,0) = \frac{1}{det\left\|c^{\alpha'}_{\beta'}\right\|} \cdot \begin{vmatrix} f^1_{y^i}(0,0)\_c^1_2 \ldots c^1_k \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ f^k_{y^i}(0,0)\_c^k_2 \ldots c^k_k \end{vmatrix} \tag{20}$$

Adding the first row of the determinant (20) to the remaining rows multiplied by $n_2,\ldots,n_k$ correspondingly and expanding the obtained determinant by the first row, we obtain:

$$\psi_{y^i}(0,0) = f^{\alpha}_{y^i}(0,0)\cdot n_{\alpha},(i=1,\ldots,l) \tag{21}$$

Let us consider the equation:

$$\psi_{\xi^1}(\xi^1, y) = 0 \tag{22}$$

Next, we will show that in a certain neighborhood of point $\xi^1 = 0, y = 0$, the equation (22) uniquely determines ξ^1 as a function of y. Substituting derivatives $\frac{\partial \xi^{\alpha'}(\xi^1, y)}{\partial \xi^1}$ determined by Cramer's rule from the system (13) into (17), derivative $\psi_{\xi^1}(\xi^1, y)$ can be represented as follows:

$$\psi_{\xi^1}(\xi^1, y) = \frac{det\left\|\frac{\partial F^i}{\partial \xi^j}\right\|}{det\left\|\frac{\partial F^{\alpha'}}{\partial \xi^{\beta'}}\right\|},(i,j=1,\ldots,k),(\alpha', \beta' = 2,\ldots,k) \tag{23}$$

Adding the first row of the determinant $det\left\|\frac{\partial F^i}{\partial \xi^j}\right\|$ to the remaining rows multiplied by

$n_2,\ldots,n_k$

correspondingly and assuming $n_1 = 1$, we obtain the resultant determinant and designate it as $\Delta(\xi^1, y)$.

Let us replace the equation (22) with an equivalent equation:

$$\Delta(\xi^1, y) = 0 \tag{24}$$

In a certain neighborhood of point $\xi^1 = 0, y = 0$, the equation (24) determines a single-valued function:

$$\xi^1 = \tilde{\xi}^1(y) \tag{25}$$

which has continuous partial derivatives. The function (25) also satisfies the equation (22). Using the Taylor series, we can express the equation (16) as follows:

$$\psi(\tilde{\xi}^1(y),y)+\frac{1}{2}\psi_{\xi^1\xi^1}(\hat{\xi}^1,y)(\xi^1-\tilde{\xi}^1(y))^2=0$$
$$\xi^1<\hat{\xi}^1<\tilde{\xi}^1(y) \tag{26}$$

Let us consider function $\psi(\tilde{\xi}^1(y),y)$ which we designate as $\chi(y)$:

$$\chi(y)=\psi(\tilde{\xi}^1(y),y) \tag{27}$$

where $\chi(0)=0$. Thus, we calculate $\frac{\partial\chi(0)}{\partial y^i}$.

$$\frac{\partial\chi(0)}{\partial y^i}=\psi_{\xi^1}(0,0)\frac{\partial\tilde{\xi}^1(0)}{\partial y^i}+\psi_{y^i}(0,0) \tag{28}$$

As long as $\psi_{\xi^1}(0,0)=0$ and considering (21), we obtain:

$$\frac{\partial\chi(0)}{\partial y^i}=f^{\alpha}_{y^i}(0,0)\cdot n_\alpha \tag{29}$$

If we assume $\frac{\partial\chi(0)}{\partial y^i}\neq 0$, then in a certain neighborhood of point $y=0$, equation $\chi(y)=0$ determines smooth surface π, which passes through this point, and when y passes through surface π, function $\chi(y)$ changes its sign. As long as $\psi_{\xi^1\xi^1}(\xi^1,y)$ maintains a constant sign which coincides with the sign of p, we can conclude that when the values of y are quite close to $y=0$ and satisfy condition $p\cdot\chi(y)\leq 0$, two functions are defined $\xi^1_{\nu^1}(y),\xi^1_{\nu^2}(y)$, which satisfy the equation (26). If $y\in\pi$, then $\xi^1_{\nu^1}(y)=\xi^1_{\nu^2}(y)=\tilde{\xi}^1(y)$.If $y\notin\pi$, then $\xi^1_{\nu^1}(y)<\tilde{\xi}^1(y)<\xi^1_{\nu^2}(y)$. Functions $\xi^1_{\nu^1}(y),\xi^1_{\nu^2}(y)$ have continuous partial derivatives, if $y\notin\pi$. Let us substitute functions $\xi^1_{\nu^1}(y),\xi^1_{\nu^2}(y)$ into functions $\tilde{\xi}^{\alpha'}(\xi^1,y)$ and obtain two solutions of the system (7):

$$\xi^1_{\nu^1}(y),\tilde{\xi}^{\alpha'}(\xi^1_{\nu^1}(y),y)$$
$$\xi^1_{\nu^2}(y),\tilde{\xi}^{\alpha'}(\xi^1_{\nu^2}(y),y) \tag{30}$$

This completes the proof.

3 Results

The main conclusions of the present paper are as follows:

- the qualitative nature of the behavior of solutions to a system of differential equations when a small parameter tends to zero over a finite period of time is studied.
- perturbations in rectangular coordinates in the form of finite polynomials in degrees of universal variables, which are selected at each stage of approximation, are determined.

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