Transformation of Rayleigh Distribution, Properties, and Application

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ABSTRACT. The need to develop the theory of statistics and its properties follows from the fact that many types of data cannot be fitted by classical distributions. This fact invites many researchers to generate new distributions, find their properties, and implement a data set to find the best distribution that can fit the data better.

In this paper, we propose special cases of Rayleigh distribution and their relationship to well-known distributions like half-logistic distribution (HLD), generalized half-logistic distribution (GHLD), and exponentiated half-logistic distribution (EHLD). We have mainly discussed the relationship of a transformation technique of those special cases of Rayleigh distribution with different parameter values to the assigned distributions (HLD, GHLD, EHLD). We also show the mathematical statistical properties of such special cases like the r^{th} moment, central moment, incomplete moments, the probability weighted moments, the stochastic ordering, and interval estimation within the proposed parameters. Consequently, such properties are derived to generate modern statistical characteristics related to the special cases of Rayleigh distribution. Moreover, we have set table for the calculations of particular cases with their derived moments that have previously found their theoretical representations. Finally, we set off some conclusions related to the results of this humble work.

1. INTRODUCTION

In this paper, we propose a special case of Rayleigh distribution with parameter function $\tau(\lambda) = \frac{1}{2\lambda}$. We show several properties of the special case like transformation and its relationship to modern distributions known by HLD, GHLD, and EHLD distributions, respectively. We also show other properties like moments, and non-Bayesian methods, interval estimation. The main interesting result of this study related to the relationship of the random variable (random variable) X that is distributed Rayleigh with parameter function $\tau(\lambda) = \frac{1}{2\lambda}$ (briefly $X \sim \mathrm{Ray}(\frac{1}{2\lambda})$) to the HLD, GHLD, and EHLD distributions with help of transformation techniques. Several different results and properties were shown in this paper with proof of them.

The importance of Rayleigh distribution has been shown in many kinds of literature that show how such distribution is powerful in analysis. The early basic form of Rayleigh distribution was derived by Lord Rayleigh (1842-1919) who employed it to describe the skewed data, [1]. Diebolt and Robert presented the deviation and distance measures in economics, and those statistical properties are appropriate to other natural life data, [2]. The extended probability density function, moments, estimations, analysis of COVID-19 data, and other important were shown in [3-5].

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Several aspects of real life have been studied by Rayleigh distribution and its extensions because of the great role of this distribution in interpreting many phenomena. Lord Rayleigh (1842-1919) presented this distribution to analyze and describe skewed data [1]. The one scale parameter of Rayleigh distribution has been studied by many authors like Robert C.P. Diebolt and Robert [6].

He has discussed deviation and distance measures in economic, which can be applied to other natural phenomena data. Lehmann [7] was the first researcher who extended the probability density function. A modern generalization of Rayleigh distribution has been presented by Kundu and Raqab and they estimated its new parameters by using various estimation methods, [6]. A derivation of a new generalization of Rayleigh distribution was presented by Voda by using a conservative technique, [7,8].

Balakrishnan and Wong (1991) have presented HLD, and prove that this distribution has similar properties to the exponential distribution, [2]. Kang et al. (2008) found the maximum likelihood estimate MLE, and approximate MLE of the scale parameter of the HLD. This was implemented on the progressive type-II censored sample. Also, Seo et al. (2017) provided a robust method to estimate the unknown parameters of two-parameters distribution on the same progressive type-II censored sample, [6]. Similarly, Kim et al. (2011) have studied the GHLD under the progressive type-II censored sample. Whereas, Kang et al. (2014), and Seo and Kang (2015b) developed the Bayesian estimation method of EHLD, and derived the moments of it, respectively, see [5].

This paper is organized as follows: In the following section, we present the basic concepts linked to the statistical concepts of Rayleigh distribution, and other related distribution that our study focuses on. Afterwards, this section also involves transformation between the desired special case and other distributions. Section three is devoted to derivations of other statistical properties like moments, estimation, and application.

When a random variable $X \sim \mathrm{Ray}(\frac{1}{2\lambda})$, then the probability density function (pdf), the distribution function, and survival functions are given by the following

$$f(x) = 2\lambda x e^{-\lambda x^2}, x > 0, \lambda > 0 \quad (1.1)$$

$$F(x) = 1 - e^{-\lambda x^2}, x > 0, \quad (1.2)$$

$$S(x) = e^{-\lambda x^2}, x > 0, \lambda > 0 \quad (1.3)$$

2. TRANSFORMATION TECHNIQUE

Using the transformation technique of random random variable X with pdf in (1.1) leads to obtaining the following distributions that are known by HLD, GHLD, and EHLD, respectively.

Theorem 2.1 Let $g(y)$ be the pdf of the random variable Y. If the random variable $X \sim \mathrm{Ray}(\frac{1}{2})$, then $Y = \frac{1}{\theta}\ln\left(2e^{x^2} - 1\right)$ is distributed HLD.

Proof: By using the pdf transformation technique, the function $Y = \frac{1}{\theta}\ln\left(2e^{x^2} - 1\right)$ assia has one to one, onto properties, and $0 \le y < \infty$. Also, the inverse function is $x = \sqrt{\ln\left(\frac{1+e^{-\theta y}}{2e^{-\theta y}}\right)}$, and the Jacobian of the desired transformation is $J = \frac{\theta}{2\sqrt{\ln\left(\frac{1+e^{-\theta y}}{2e^{-\theta y}}\right)}\,1+e^{-\theta y}}$.

Therefore, $g(y) = f\left(u^{-1}(x)\right).J = \frac{2\theta e^{-\theta y}}{\left(1+e^{-\theta y}\right)^2}, 0 \le y < \infty$, and $\theta > 0$, which is the pdf of HLD distribution.

Theorem 2.2 Let $g(y)$ be the pdf of the random variable Y. If the random variable $X \sim \mathrm{Ray}(\frac{1}{2\lambda})$, then the random variable $Y = \frac{1}{\theta}\ln\left(2e^{x^2} - 1\right)$ is distributed GHLD with parameters θ, λ.

Proof: Similarly, we obtain the same inverse and Jacobian in Theorem 2.1. Therefore,

$g(y) = f\left(u^{-1}(x)\right).J = \theta\lambda\left[\frac{2e^{-\theta y}}{1+e^{-\theta y}}\right]^{\lambda}\frac{1}{1+e^{-\theta y}}, 0 \le y < \infty$, and $\theta > 0$, which is the pdf of GHLD distribution.

Theorem 2.3 Let $g(y)$ be the pdf of the random variable Y. If the random variable $X \sim Ray(\frac{1}{2\lambda})$, then the random variable $Y = \frac{1}{\theta}\ln\left(\frac{e^{x^2}+1}{e^{x^2}-1}\right)$ is distributed EHLD with parameters θ, λ.

Proof: The inverse, and the Jacobian of the given function y are, respectively

$x = \sqrt{\ln\left(\frac{1+e^{-\theta y}}{2e^{-\theta y}}\right)}$, and $|J| = \frac{2\theta e^{-\theta y}}{2\sqrt{\ln\left(\frac{1+e^{-\theta y}}{2e^{-\theta y}}\right)}\,1-e^{-2\theta y}}$

Therefore, $g(y) = f(u^{-1}(x)).|J| = \theta\lambda\left[\frac{1-e^{-\theta y}}{1+e^{-\theta y}}\right]^{\lambda}\frac{2e^{-\theta y}}{1-e^{-2\theta y}}, 0 \le y < \infty$, and $\theta > 0$, which is the pdf of EHLD distribution.

3. PROPERTIES, and ESTIMATION

3.1 MOMENTS

In this part, we show the derivations of r^{th} moment, the first, the second moment, and the variance of the random variable $x \sim Ray\left(\frac{1}{2\lambda}\right)$.

Theorem 3.1 If the random variable $X \sim Ray(\frac{1}{2\lambda})$, then the r^{th} moment about the origin is

$$E(X^r) = \frac{\Gamma\left(\frac{r}{2}+1\right)}{\lambda^{\frac{r}{2}}}, r = 1,2, \ldots \quad (3.1)$$

Proof Let $u = x^2$.implies that, when $x = 0$, and ∞, then $u = 0$, and ∞, respectively. Then

1. $E(X^r) = 2\lambda\int_0^{\infty} x^r x\, e^{-\lambda x^2} dx$

And $x = u^{\frac{1}{2}}$, implies the $dx = \frac{1}{2}u^{-\frac{1}{2}}du$. Thus

$$E(X^r) = \lambda\int_0^{\infty} u^{\frac{r}{2}} e^{-\lambda u} du = \lambda\int_0^{\infty} u^{\frac{r}{2}+1-1} e^{-\lambda u} du$$
$$= \lambda\,\Gamma\left(\frac{r}{2}+1\right)\lambda^{-\frac{r}{2}-1}\int_0^{\infty}\frac{1}{\Gamma\left(\frac{r}{2}+1\right)\lambda^{-\frac{r}{2}-1}}u^{\frac{r}{2}+1-1}e^{-\lambda u}du$$

It is clear that $U \sim G\left(\frac{r}{2}+1, \frac{1}{\lambda}\right)$. Thus

$$\int_0^{\infty}\frac{1}{\Gamma\left(\frac{r}{2}+1\right)\lambda^{-\frac{r}{2}-1}}u^{\frac{r}{2}+1-1}e^{-\lambda u}du = 1$$

Therefore, $E(X^r) = \lambda\,\Gamma\left(\frac{r}{2}+1\right)\lambda^{-\frac{r}{2}-1} = \frac{\Gamma\left(\frac{r}{2}+1\right)}{\lambda^{\frac{r}{2}}}$

In particular, when $r = 1,2$, we can obtain that the 1st, 2nd moments, and variance of the random variable $X \sim Ray\left(\frac{1}{2\lambda}\right)$ are, respectively

1. $E(x) = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}$;
2. $E(X^2) = \frac{1}{\lambda}$;
3. $\sigma^2 = var(X) = \frac{4-\pi}{4\lambda}$;

By using the formula in (3.1), and $\lambda = 0.5(0.5)4$, then the mean and variance are calculated as shown in Table…We can notice that when the mean, the second expectation, and the variance decrease as the shape parameter λ increases.

Table 3.1 1st moment, 2nd moment, and variance with various parameter's values

λ	0.5	1	1.5	2	2.5	3	3.5	4
$E(X)$	1.253316	0.886228	0.723602	0.626658	0.5605	0.511664	0.473709	0.443114
$E(X^2)$	2	1	0.666667	0.5	0.4	0.333333	0.285714	0.25
$Var(X)$	0.4292	0.2146	0.143067	0.1073	0.08584	0.071533	0.061314	0.05365

We can also derive other moments like, the coefficient of skewness (γ_1), and the coefficient of kurtosis (γ_2), and obtain their forms that are

1. $\gamma_1 = \frac{\pi^{\frac{3}{2}} - 3\pi^{\frac{1}{2}}}{(4-\pi)^{\frac{3}{2}}}$;
2. $\gamma_2 = \dfrac{32 - 24\pi - 3\pi^2}{(4-\pi)^{\frac{3}{2}}}$

Theorem 3.2 If the random variable $X \sim Ray(\frac{1}{2\lambda})$, then the rth central moment about the mean is

$$E\left[(X - E(X))^r\right] = \sum_{i=0}^{r} \binom{r}{i} \left(-E(X)\right)^i \frac{\Gamma\left(\frac{r-i}{2} + 1\right)}{\lambda^{\frac{r-i}{2}}}, r = 1,2, \dots$$

Proof Similar to the prove of theorem 3.1, suppose that $u = x^2$, implies that, when $x = 0$, and ∞, then $u = 0$, and ∞, respectively, and by using the binomial expansion formula, we obtain that

$$E\left[(X - E(X))^r\right] = E\left[\sum_{i=0}^{r} \binom{r}{i} \left(-E(X)\right)^i X^{r-i}\right] = \sum_{i=0}^{r} \binom{r}{i} \left(-E(X)\right)^i E(X^{r-i}) \text{ , } r = 1,2, \dots$$

And, for the random variable $X \sim Ray(\frac{1}{2\lambda})$, the moment $E(X^{r-i})$ can be found as follows

$$E(X^{r-i}) = 2\lambda \int_0^\infty x^{r-i} x \, e^{-\lambda x^2} dx$$

Similar to the proof of $E(X^r)$, we can see that $U \sim G\left(\frac{r-i}{2} + 1, \frac{1}{\lambda}\right)$

Therefore,

$$E\left[(X - E(X))^r\right] = \sum_{i=0}^{r} \binom{r}{i} \left(-E(X)\right)^i \frac{\Gamma\left(\frac{r-i}{2} + 1\right)}{\lambda^{\frac{r-i}{2}}}, r = 1,2, \dots$$

Moreover, we can define the differential entropy of the random variable X that has the pdf in (1.1), [4].
$H(f(x)) = -\int_{-\infty}^{\infty} f(x) \log(f(x)) dx$

Theorem 3.3 If the random variable $X \sim Ray\left(\frac{1}{2\lambda}\right)$, then its differential entropy is

$H(f(x)) = 1 - \ln(2\lambda) - \frac{1}{2}(\ln(4\lambda) - \gamma)$,

where $\gamma \cong 0.5772$ is well-known as Euler's constant.

The proof follows directly from the proof of $E(\ln x)$ which is shown in [4].

Theorem 3.4 Let the random variables X, and Y have respectively, the pdf's $f_1(x), f_2(x)$, where $X \sim Ray\left(\frac{1}{2\lambda_1}\right)$, and $Y \sim Ray\left(\frac{1}{2\lambda_2}\right)$. If $\lambda_1 \le \lambda_2$, then $X \le Y$, and they are stochastic ordering.

Proof The pdf of the random variables X, and Y are, respectively

$f_1(x) = 2\lambda_1 x e^{-\lambda_1 x^2}$, $f_2(x) = 2\lambda_1 x e^{-\lambda_1 x^2}$

Hence, the proportional be $\mathbf{f_1(x)}$, and $\mathbf{f_2(y)}$ is

$$\frac{f_1(x)}{f_2(x)} = \frac{\lambda_1}{\lambda_2} e^{-x^2(\lambda_1 - \lambda_2)} \le 1$$

Therefore, $X \le Y$.

Theorem 3.5 If the random variables $X \sim \mathrm{Ray}\left(\frac{1}{2\lambda_1}\right)$ is defined over a subset of a set of real numbers, then the r^{th} incomplete moment is

$$E(X^r) = \frac{\lambda\left(\frac{r}{2}!\right)}{\lambda^{\frac{r}{2}+1}}\left[1 - e^{-\lambda t}\sum_{i=0}^{n}\frac{(\lambda t)^i}{i!}\right]$$

Proof by using the formula of incomplete moments, that is

$$E(X^r) = \int_0^t x^r\, 2\lambda x e^{-\lambda x^2} dx$$

make a change of variables, we obtain that

$$E(X^r) = \lambda\int_0^t u^{\frac{r}{2}}\, e^{-\lambda u} du = \frac{\lambda\left(\frac{r}{2}!\right)}{\lambda^{\frac{r}{2}+1}}\left[1 - e^{-\lambda t}\sum_{i=0}^{n}\frac{(\lambda t)^i}{i!}\right]$$

This completes the proof.

Theorem 3.6 If $X \sim \mathrm{Ray}\left(\frac{1}{2\lambda}\right)$, then the r^{th}probability-weighted moment has the following form

$$E[X^r F^s(X)] = \lambda^{\frac{-r}{2}}\Gamma\left(\frac{r}{2}+1\right)\sum_{j=0}^{\infty}\binom{s}{j}(-1)^j\frac{1}{(1+j)^{\frac{r}{2}+1}}$$

Proof by recalling the formula of PWM in [9], we have

$$E[X^r F^s(X)] = \int_0^{\infty} x^r F^s(x) f(x) dx = 2\lambda\int_0^{\infty} x^r\left(1 - e^{-\lambda x^2}\right)^s x e^{-\lambda x^2} dx$$

By changing the variables, we obtain that

$E[X^r F^s(X)] = \lambda\int_0^{\infty} u^{\frac{r}{2}}\left(1 - e^{-\lambda u}\right)^s e^{-\lambda u} du$

By using the binomial series expansion, it is clear that

$$\left(1 - e^{-\lambda u}\right)^s = \sum_{0}^{\infty}\binom{s}{j}(-1)^j\left(e^{-\lambda u}\right)^j$$

Thus, $E[X^r F^s(X)] = \lambda\sum_0^{\infty}\binom{s}{j}(-1)^j\int_0^{\infty} u^{\frac{r}{2}+1-1} e^{-u\lambda(1+j)} du = \lambda^{\frac{-r}{2}}\Gamma\left(\frac{r}{2}+1\right)\sum_{j=0}^{\infty}\binom{s}{j}(-1)^j\frac{1}{(1+j)^{\frac{r}{2}+1}}$

This completes the proof

3.2 ESTIMATION

We estimate the shape parameter (λ) of X from Rayleigh distribution by using maximum likelihood method, and also show its confidence interval.

Consider that $X_1, \ldots, X_n$ are random sample (independent and identically distributed) from Rayleigh with parameter $\frac{1}{2\lambda}$. Then, the likelihood of the parameter λ from $x \sim \mathrm{Ray}\left(\frac{1}{2\lambda}\right)$ is

$$L(\lambda, \underline{x}) = \prod_{i=1}^{n} 2\lambda x_i e^{-\lambda x_i^2} = 2^n\lambda^n\prod_{i=1}^{n} x_i e^{-\lambda\sum_{i=1}^{n} x_i^2}$$

And the corresponding log-likelihood is

$$\ell = \ln L(\lambda, \underline{x}) = n\ln 2 + n\ln\lambda + \sum_{i=1}^{n}\ln x_i - \lambda\sum_{i=1}^{n} x_i^2$$

by differentiating the log-likelihood with respect to λ, we obtain that

$$\frac{d\ell}{d\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}x_i^2=0$$

Then the maximum likelihood estimate (MLE) of parameter λ is

$$\hat{\lambda}=\frac{n}{\sum_{i=1}^{n}x_i^2}=\frac{n}{(n-1)S^2+n\overline{X}^2}$$

The MLE $\hat{\lambda}$ has the following expectation

$$E(\hat{\lambda})=E\left(\frac{n}{(n-1)S^2+n\overline{X}^2}\right)=\frac{n}{(n-1)E(S^2)+nE\left(\overline{X}^2\right)}=\lambda$$

which means that $\hat{\lambda}$ is unbiased estimator of λ, where $E(S^2)=\sigma^2=\frac{4-\pi}{4\lambda}$, $E\left(\overline{X}^2\right)=\frac{2-\pi}{2\lambda}$.

To establish the confidence interval (CI) for $\hat{\lambda}$, we find the second derivative with respect to λ. Then

$$\frac{d^2\ell}{d\lambda^2}=\frac{-n}{\lambda^2}$$

And by using the fact that follows from the Fisher information matrix, we obtain that

$$-E\left(\frac{d^2\ell}{d\lambda^2}\right)=\frac{n}{\hat{\lambda}^2}=\mathrm{Var}(\hat{\lambda})$$

Therefore, for the large sample and with help of the central limit theorem (C.L.T) an approximate $100\%(1-\alpha)$ CI of the parameter λ is given by

$$\left(\hat{\lambda}-Z_{1-\frac{\alpha}{2}}\sqrt{\mathrm{Var}(\hat{\lambda})},\hat{\lambda}+Z_{1-\frac{\alpha}{2}}\sqrt{\mathrm{Var}(\hat{\lambda})}\right)$$

Whereas, for small sample of the non-normal case, the approximate $100\%(1-\alpha)$ CI of that parameter is

$$\left(\hat{\lambda}-t_{1-\frac{\alpha}{2}}\sqrt{\mathrm{Var}(\hat{\lambda})},\hat{\lambda}+t_{1-\frac{\alpha}{2}}\sqrt{\mathrm{Var}(\hat{\lambda})}\right)$$

where, $Z_{\frac{\alpha}{2}}$, $t_{\frac{\alpha}{2}}$ are the standard normal distribution value, and student distribution value, respectively.

Note that the CI based on the MLE for the random variable $X\sim \mathrm{Ray}\left(\frac{1}{2\lambda}\right)$ yields lower and upper limits, respectively, and this guarantees that the parameter is nonnegative.

4. CONCLUSION

This paper has shown various properties of the special case of Rayliegh distribution with shape parameter $\left(\frac{1}{2\lambda}\right)$. Three transformation types from such distribution to important distributions HLD, GHLD, and EHLD have been shown. Various statistical properties like moments have also been shown with close forms and such proposal is not presented before to the best of our knowledge. New forms of some special moments were found, for example, the incomplete moments, the probability-weighted moments. The MLE of the proposed parameter model have been made and constructed the approximate of CI of that parameter based on that method. We show that the estimated parameter $\hat{\lambda}$ is unbiased. The results in Table 3.1 are shown that moments are decreasing by increasing the value of the parameter.

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