m-rnc rings

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Abstract: In this article, an element w of an associative ring R is called m-regular nil clean or m-rnc if expressed as $w = a^m + b$ where a^m is m-regular element and b is a nilpotent element. R is named m-regular nil clean ring or m-rnc ring. If all the elements of a ring $\mathcal{R}$ are m-rnc, some characteristics and basic properties of m-rnc rings are presented in this work.

1 Introduction

Following [1], [2], [3], [4] a ring R is called clean if all elements w in R it can be expressed as a sum of a unity $u \in$ R and an idempotent $e \in$ R similarly Diesl and Alexander [3] developed the nil clean ring, which told that R be nil clean when all elements in R can be expresses as a sum of an idempotent $e \in$ R and nilpotent $b \in$ R . A ring R is considered to be m- regular ring if and only if there is an element y in R for each a in R and a positive integer $m \geq 1$ such that, $a^m = a^m y a^m$[5], [6] when m=1,that is a= a y a, R is consider to be von Neumann. regular ring (or simply regular)[5], [6], [7] generally, all rings will be unitary and associative. For any ring, N(R), U(R), J (R), Y(R), $Idem$(R), $m - reg$ (R) , and C(R) are used to denote the set of all nilpotent elements, the group of unities $element$, Jacobson radical element of R, the right singular ideal of R, the set of the idempotent element in R, the set of m-regular elements of R and the center of R respectively. also, a ring R is denoted $NJ - ring$ if N(R) $\subseteq J$ (R) [8]. R named reduced when N(R) = {0}, [9]. Finally McCoy in [5] called a ring R to be π-regular, if $\forall\ a \in$ R, $\exists$ positive integer n and $b \in$ R, $so\ a^n = a^n b a^n$.

2 m- regular nil-clean rings

The definition of m-regular nil clean element and m-regular nil − clean rings and their characteristics also basic properties are presented in this section.

Definition 2.1

The element $x \in R$ is m-regular nil-clean or (m-rnc) if $x = a^m + b\ where\ a^m \in m\, reg$(R) $for\ a\ fixed\ positive\ integer\ m \geq l\ and\ b \in N$(R). R is named m-regular nil-clean ring, or (m-rnc ring) if every element in R is m-rnc.

Clearly m-regular rings and nil clean rings are m-rnc rings. However, in general, m-rnc rings may not be nil clean, Z_6 is m-rnc but not nil clean since 2', 5' cannot $be\ written\ as\ a\ sum\ of\ nilpotent\ and\ idempotent\ of\ Z_6$, also Z_8 is m-rnc ring for $m = 3$, but not regular.

In the following some basic properties for m-rnc element are considered.

Preposition 2.2

The element x in R is m- rnc if and only if $1 - x$ is m- rnc element.

Proof:

Assume x is m-rnc element, then $x = a^m + b$, where a^m is m-regular element for a fixed integer $m \geq 1$ and b is a nilpotent element in R then,

$1 - x = 1 - a^m - b = -a^m + (1 - b). Now, -a^m = -(a^m b a^m\ )$

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$= (-a^m)(-b)(-a^m)$ *since* $(-a^m)$ *is* $m-$ reg(R) *element. But* $(1-b^n) = 1$
$= (1-b)(1+b+b^2+\ \dots + b^{n-1})$ *thus* $(1-b)$ *is unit, hence* $(1-b)$ *is regular.*
Therefore $1-x$ *is* $m-rnc$ *element* $\rightarrow (1-b)$ *is regular* $\rightarrow m$ *regular there fore*
$1-x$ *is* $m-rnc$.
Conversely : Suppose that $(1-x)$ is m- rnc element, *so* $1-x = a^m + b$, where
$a^m \in m-reg$(R) *and* $b \in N$(R). *Hence* $-x = a^m + (b-1)$ *and*
then $x = -a^m + (b-1)$. *Therefore x is* $m-rnc$ *element.*■

Preposition 2.3

Let $\mathcal{R}$ be an abelian ring, if $x \in \mathcal{R}$ is m-rnc element, then x^n is m-rnc element *for some positive integer n if and only if* x is m-rnc element.

Proof:

If x is m-rnc element. Thus $x = a^m + b$, where a^m is m-rnc for a fixed *positive integer*, $m \geq 1$ *and* $b \in N$(R). Now, we prove x^n is m-rnc by mathematical induction;

$x^2 = (a^m+b)^2 = a^{2m} + 2a^m b + b^2 = a^{2m} + b(2a^m + b)$.

Assume that for n = k -1, the assumption is true, that is

$$x^{\kappa-1} = (a^m+b)^{\kappa-1} = \sum_{n=0}^{\kappa-1} \binom{\kappa}{n} (a^m)^n b^{\kappa-n}$$

$$= (a^m)^{\kappa-1} + \frac{(\kappa-1)}{1!}(a^m)^{\kappa-2}\, b + \frac{(\kappa-1)(\kappa-2)}{2!}$$

$$(a^m)^{\kappa-3}\, b^2 + \ \dots\dots + \frac{(\kappa-1)\,(\kappa-2)(\kappa-3)}{3!}(a^m)^{\kappa-4}\, b^3 + \ \dots\ + b^{\kappa-1}$$

We must prove the assumption is true when $n = k$,

$$x^n = x\ x^{n-1}$$

$$(a^m+b)$$

$$\left[(a^m)^{\kappa-1} + \frac{\kappa-1}{1!}(a^m)^{\kappa-2}\, b + \frac{(\kappa-1)(\kappa-2)}{2!}(a^m)^{\kappa-3}\ b^2 + \dots + b^{\kappa-1}\right]$$

$$= (a^m)^{\kappa} + b$$

$$\left[(a^m)^{\kappa-1} + \frac{\kappa(\kappa-1)}{2!}(a^m)^{\kappa-2}\, b + \frac{\kappa(\kappa-1)(\kappa-2)}{3!}(a^m)^{\kappa-3}\ b^2 \dots + b^m\right]$$

Suppose that $l = b$

$$\left[(a^m)^{\kappa-1} + \frac{\kappa(\kappa-1\,)}{2!}(a^m)^{\kappa-2}\ b + \frac{\kappa(\kappa-1)(\kappa-2)}{3!}(a^m)^{\kappa-3}\ b^2 \dots + b^{\kappa}\right]$$

$\in N$(R) Therefore $x^n = (a^m)^n + l$

Now, if x^n is m-rnc element, to prove x m-rnc element, then $x^n = a^m + b$ *so* $x^n - a^m = b$. Now $x^n - a^m = (x-a^m)(a^m + a^m x + a^m x^2 + \dots . + x^{n-1}) \in N$(R). *Hence* $x - a^m \in N$(R),
thus $x - a^m = b_1$ *so* $x = a^m + b_1$, *therefore x is* $m-rnc$ *element.*■

Proposition 2.4

Let R be an abelian ring with $2 \in N$(R) and x is m-rnc element. Then $x^2 - x$ it will be m-rnc element.

Proof:

If x be m- rnc element, there exists $a^m \in m-reg$(R) for fixed integer $m \geq 1$,
and $b \in N$(R), so that $x = a^m + b$. *Now*, $x^2 = (a^m+b)^2 = a^{2m} + 2a^m b + b^2$.
Hence $x^2 - x = (a^{2m} + 2a^m b + b^2) - (a^m + b) = (a^{2m} - a^m) + b(b-1+2a^m)$
$= a^m(a^m - 1) + b(b-1+2a^m)$. There for $x^2 - x$ *is* $m-rnc$ *element.*■

Preposition: 2.5

Every homeomorphic surjective image of m-rnc ring is m-rnc ring.

Proof:

Let $\emptyset$:R→ H is homomorphism function from m-rnc ring R onto the ring H, let $x \in$R and $h \in H$ so $\emptyset(x) = h$, thus we can write $x = a^m + b$ for some $a^m \in m-reg$(R) and $b \in N$(R), then $h = \emptyset(x) = \emptyset(a^m+b) = \emptyset(a^m) + \emptyset(b)$ since $a^m \in m-reg$(R), that is $a^m = a^m y a^m$ for some $y \in$R *and a fixed positive integer* $m \geq 1$ and since, $b \in N$(R), so
$b^n = 0$, where n$\in z^+$. Now $h = \emptyset(a^m y a^m) + \left(\emptyset(b)\right)^n = \emptyset(a^m)\emptyset(y)\emptyset(a^m) + \emptyset(b^n)$

$= (\emptyset(a))^m \emptyset(y)(\emptyset(a))^m + \emptyset(0), supoose\ that\ \emptyset(a) = r \in H\ and\ \emptyset(b) = b^*, hence\ h = r^m \emptyset(y) r^m + b^*\ \ is\ m - rnc\ element\ in\ H. There\ fore\ H\ is\ m - rnc\ ring.$■

Proposition 2.6

Suppose R is a ring with I ideal over R. So R /I is m-rnc when R is m-rnc.

Proof :-

Since R is m-rnc ring and let I be an ideal of R so $x + I \in$ R /I , where $x \in$ R, then $x = a^m + b$, when $a^m \in$ m-reg(R) for a fixed positive integer m and b∈N(R).

$x + I = b + a^m + I) = (b + I) + (a^m + I)$
$= (b + I) + (a^m \ell a^m + I), where\ \ell \in$R
$= (a^m + I)(\ell + I)(a^m + I) + (b + I)$
$a^m + I\ is\ m - regular\ element\ in$ R$/I$
$since\ b \in N$(R)$, then\ b + I\ is\ nilpotent\ element\ in$ R$/I$ $Therefore$ R$/I$ $is\ m - rnc\ ring.$■

Proposition 2.7

Let N a nil ideal of a R. So R is m-rnc if R /N is m-rnc ring.

Proof:-

Suppose R /N is m-rnc ring and $x + N \in$R/N$, then\ x + N = (b + N) + (a^m + N)$ so, $x + N = a^m + N,\ thus\ x - a^m \in N\ that\ is\ \ x - a^m = b^*N, it\ \ follows\ that$
$x = a^m + b^*\ Therefore\ R\ is\ m - rnc\ ring.$■

Proposition 2.8

Assume R be NJ ring and $r(\ x\ ) \subseteq$ N(R). Such that R is m- rnc ring, if and only if, R /$r(x)$ is m-rnc ring.

Proof:

If R is m-rnc ring, so R /$r(x)$ is also by Proposition 2.5. conversely :- suppose that R /$r(x)$ is m-rnc ring and let $x + r(x) \in$R/r(x) where $x \in$R then there exists $a^m \in m\ reg$(R) for a fixed integer $m \geq 1$ and $b \in N$(R) such that $x + r(x) = (a^m + r(x)) + (b + r(x))$, it follows that $x + r(x) = (a^m + b) + r(x), which\ imples\ that\ (x - (a^m + b\ )) \in r(x), since$ r(x) $\subset\ N$(R),then
$(x - a^m)\ - b \in N$(R)$. Thus\ (x - a^m) is\ nilpontent\ element. Therefor$ R $is\ m -\ rnc\ ring.$■

proposition 2.9

Let R be m-rnc ring and $x \in$R, if xR not contain zero idempotent. Then x is a sum of right unit element and nilpotent element.

Proof

Assume that xR $contains\ no\ non\ zero\ idempotent$, select $a^m \in$ m-reg(R) for a $fixed\ positive\ integer.\ m \geq 1$ and $b \in N$(R) such that $x - 1 = a^m + b$. Then $x = a^m + (b + 1)$, where $(b + 1)\ is\ unit\ element\ and\ a^m = a^m y\ a^m\ for\ a\ fixed\ positive\ integer\ m \geq 1$ and for some $y \in$R. Since $xy\ a^m = (a^m + b)ya^m$, then $x(1 - ya^m) = (b + 1)(1 - ya^m)$ so,
$(b + 1)((1 - ya^m)(b + 1)^{-1} = x\ (1 - ya^m)(b + 1)^{-1} \in x$R, since clearly $(b + 1)(1 - ya^m)(b + 1)^{-1}$ is idempotent in the ring R. by assumption, $(b + 1)\ (1 - ya^m)(b + 1)^{-1} = 0$.
$Hence$,$1 - ya^m = 0\ and\ then\ ya^m = 1$, follows that$(a^m)$ is unit. Therefor x is the sum of a right unity and a nilpotent.■

proposition 2.10

If R an abelian m-rnc ring. then e R e is m- rnc ring.

Proof :-

Let $x \in e$ R $e \subseteq$R. then there exists $a^m \in m - reg$(R) for a fixed integer $m \geq 1$ and $b \in N$(R), such that $x = a^m + b\ now, exe = ea^m e + ebe, e \in Idem$ (R), since
$a^m \in$ m − reg(R), then exists $c \in$R such that, $a^m = a^m\ c\ \ a^m$ and, $e\ a^m e = e\ a^m c\ a^m\ e = e\ a^m c\ e\ a^m$, thus $e\ a^m e\ is\ m - reg$(R) element in e R e, also $(e\ b\ e)^n = e^n b^n e^n = 0$. For some n∈ z^+ that is $e\ b\ e$ is nilpontent element $in\ e$ R e, thus $x = (a^m e). c. (a^m e) + ebe$ and therefor , e R $e\ is\ m - rnc$ ring.■

Preposition 2.11 :

Suppose that R be m-rnc ring .Then J(R) is nilpotent ideal.

Proof :

$let\ x \in J$(R)$, since$R $is\ m - rnc\ ring\ , then\ x = a^m + b\ where\ a^m \in m -\ reg$(R)
$for\ a\ fixed\ integer\ m \geq 1and\ b \in N(R). Now\ , we\ must\ prove\ that\ a^m \in N$(R),
$a^m = a^m c\ a^m, for\ some\ c \in$R $, it\ follows\ that\ (1 - a^m c)a^m = 0$

$since\ x\ \in J(\mathrm{R}), then\ a^m\ \in\ J(\mathrm{R}) and\ a^m c\ \in\ J(\mathrm{R}), hence\ (1-a^m c)$
$is\ unit$ [8]. $Then\ there\ exists\ u\ \in\mathrm{R}\ such\ that\ u(1-a^m c)=1$
$multiply\ by\ a^m from\ the\ right, we\ get\ u(a^m - a^m c\ \ a^m\ ) = a^m,$
$that\ is\ a^m = 0. Therefore\ x\ is\ nilpotent\ element\ and\ J(\mathrm{R}) is\ \ nilpontent\ \ ideal$.■

corollary 2.12

Assume R a reduced m-rnc ring. So J(R)=(0).

proof :

Let J(R)$\neq$ (0), thus exists non-zero element $x \in J(\mathrm{R})$, by proposition (2.11) we have x is nilpotrnt elemen. since R is reduced ring, then $x = 0$. Hence $J(\mathrm{R})=0$.

Preposition 2.13 :

If R $is\ m-rnc$ ring , then $y(\mathrm{R})$is nilpotent ideal.

Proof

Suppose that $x \in y(\mathrm{R})$, that $x\ is\ m-rnc$ element , it follows that $x = a^m + b$, $a^m\ \in m-reg(\mathrm{R})$ such that $a^m =\ a^m c a^m$for a fixed integer $m \geq 1$ and $c\ \in\mathrm{R}\ \ and\ b \in N(\mathrm{R})$. Now to prove x is nilpoent, since $x \in y(\mathrm{R})$, implies that , $a^m c\ \in\ Y(\mathrm{R})$, this mean that $r(a^m c)$is right singular ideal in the ring R, let $r(a^m c) \cap a^m\mathrm{R} \neq 0$, $Then\ there\ exists\ 0 \neq y \in\ \ r(a^m b) \cap a^m\mathrm{R}$ and weobtain $a^m cy = 0\ and\ y = a^m r = a^m c a^m r\ = a^m cy = 0$, where $r\ \in\mathrm{R}$, that is $r(a^m c) \cap a^m\mathrm{R}= 0$, since $r(a^m c)$is right singularideal, then $a^m\mathrm{R}\ = 0$, thus $a^m = 0$. $Hence\ x = 0 + b$ is nilpotent element and therefore $y(\mathrm{R})$is nil potent ideal .■

Proposition 2.14

Suppose that $\mathcal{R}$ be a ring so that m-reg ($\mathcal{R}$) ⊆ C($\mathcal{R}$). If $\mathcal{R}$ is m-rnc ring. then C ($\mathcal{R}$) is m-rnc ring.

proof :-

Let $\mathcal{R}$ is m-rnc ring, and assume that $x\ \in \mathrm{C}\ (\mathcal{R})$, then x can be write as; $x = a^m +\ b$, where $a^m is\ m-reg(\mathcal{R})$ for a fixed positive integer $m \geq 1\ and\ \ b \in\ N(\mathcal{R})$, that is $a^m = a^m y\ a^m$, by assume $\ell = y\ a^m y$, that is $\ell = y a^m y a^m y$, multiply by a^m, implies that $a^m\ \ell\ a^m = a^m\ y\ a^m\ y\ a^m\ y$, hence $a^m\ \ell\ a^m = a^m y a^m = a^m$, thus $a^m \ell\ a^m \in m-reg(\mathcal{R}) \subseteq \mathrm{C}(\mathcal{R})$, which implies that b =$(x - a^m) \in \mathrm{C}\ (\mathcal{R})$.
There fore C($\mathcal{R}$)is $m-rnc$.□

Proposition 2.15

For R abelian ring and P is the primitive ideal, and R /P be m-rnc ring. Then P is the maximal ideal in R.

Proof:

Let $x \in$ R, thus $x\ +\ \rho\ \in$ R /P, since R /P is m-rnc ring, so there exists $a^m\ \in$ m-reg(R) for a fixed positive integer $m\ \geq 1$ and $b \in N(\mathrm{R})$ such that ;

$x\ +\ \rho\ =\ (a^m +\ b)\ +\ \rho\ =\ (a^m\ y\ a^m\ +\ b)\ +\rho$, for some $y\ \in$R, it follows that

$x\ +\rho = (a\ +\rho)^m(y\ +\rho)\ (a\ +\ \rho)^m\ +\ (b\ +\rho)$
$=\ (a^m\ +\ \rho)\ (y\ +\rho)\ (a^m\ +\ \rho)\ +\ (b\ +\rho)$
$=\ (a^m\ y\ a^m\ +\rho)\ +\ (b\ +\ \rho)$

Hence $x - (a^m\ y\ a^m\ +\ b) \in \rho$, that is $(a^m\ +\ b)$ - $(a^m\ y\ a^m\ +\ b)\ \in\ \rho$, implies that, $a^m - a^m y a^m\ \in\ \rho$ and thus $a^m\ (1 - y a^m)\ \in\ \rho$, assume that $a^m\ \notin\ \rho$, then $(1 - y a^m)^n \in \rho$ for some positive integer n.

Now $(1 - y a^m)^n = 1 - [\sum_{\kappa-1}^{n} c_\kappa^n\ (-1)^{\kappa-1}\ y^\kappa\ a^{m(\kappa-1)}] a^m \in \rho$ where $c_\kappa^n = \frac{n!}{\kappa!(n-\kappa)!}$
and let $z = \sum_{\kappa=1}^{n} c_\kappa^n\ (-1)^{\kappa-1}\ y^k\ a^{m(\kappa-1)}, then\ 1 - z a^m\ \in \rho$, it follows,
$1 + \rho = (z + \rho)(a^m + \rho) = (z a^{m-1} + \rho)(a + \rho)$ hence $a\ +\ \rho$ has inverse, thus R/P is the division ring, and so that P is the maximal ideal of R ■.

Proposition 2.16

Let R a ring and $r(x^{m+1}) \subseteq\ r(x^m)$, for all $x \in$ R and positive integer $m \geq 1$, then R is m-rnc ring if R /r (a) is m-rnc ring.

Proof :

Suppose that R /r (a) is m-rnc ring, thus $x + r(a)\ \in\mathrm{R}/r(a)$, where $x \in$ R and $x =\ a^m + b$ for $a^m \in\ m-reg(\mathrm{R})$ for a fixed positive integer $m \geq 1$ and $b\ \in\ N$ (R) so that.

$x + r(a) = (a^m + b) + (a)$
$= (a^m + r(a) + (b + r(a))$

$= (a^m ya^m + r(a)) + (b + r(a))$
$= (a^m ya^m + b) + r(a), which\ implies\ that (a^m + b) + r(a)) = (a^m\ ya^m + r(a)) + (b + r(a))$ that is $(a^m + b) - (a^m\ ya^m + b)) \in r(a)$ it follows that
$a^m - a^m ya^m \in r(a), that\ is\ a\ (a^m - a^m ya^m) = 0$ then $a^{m+1}\ (1 - ya^m) = 0$,
Hence $(1 - y\ a^m) \in r\ (a^{m+1}) \subseteq r\ (a^m)$, implies that $a^m(1 - y\ a^m) = 0$
We obtain that, $a^m = a^m ya^m$, therefore R is m-rnc ring.■

Proposition 2.17
A direct product R = $\pi_{i\in I}$ R_i of rings $\{R_{i\in I}\}_{i\in I}$ is m-rnc ring if and only if R_i is m-rnc ring for each i.
Proof:
Let R is m-rnc ring, thus each of $\{R_i\}_{i\in I}$ is m-rnc ring by proposition 2.5 Conversely, assume that each of R_i for every i and set $x = (x_i)_{i\in I} \in \pi_{i\in I}$ R_i. For each i, write $x_i = a_i^m + b_i$ where $a_i^m \in$ m-reg(R) for a fixed positive integer $m \geq 1$ and $b_i \in N(R_i)$.
since $a_i^m \in$ m-reg(R), there exists $y_i \in R_i$ such that $a_i^m\ y_i\ a_i^m = a_i^m$, thus $x = (a_i^m)_{i\in I} + (b_i)_{i\in I}$ Where $(a_i)_{i\in I} \in$ m-reg($\pi_{i\in I}$ R_i) and $(b_i)_{i\in I} \in N\ (\pi_{i\in I}\ R_i)$ Therefore $\pi_{i\in I}$ R_i *is* $m - rnc\ ring$.■

Proposition 2.18
Suppose R is a ring in which 2 is invertible. Then R is m-rnc , if and only if, every element of R is the sum of m-reg(R) and unite.
Proof:
Let R is m-rnc ring and x is any *element* of R , thus $\frac{x+1}{2} \in$R, follows that $\frac{x+1}{2} = a^m + b$ where $a^m \in$ m-reg(R) for a fixed positive integer $m \geq 1$ and $b \in N$(R), implies that
$x = 2a^m + (2b - 1)$, hence there exists $y \in$R, such that $a^m ya^m = a^m$. Thus $(a^m + a^m)\frac{y}{2}\ (a^m + a^m) = \frac{a^m ya^m}{2} + \frac{a^m ya^m}{2} + \frac{a^m ya^m}{2} + \frac{a^m ya^m}{2} = \frac{1}{2}(a^m + a^m + a^m + a^m) = 2a^m$.
Thus $2a^m$ m-reg(R) and $(2b - 1) \in U$(R), such that x is a sum of is m-reg (R) element and a unite.
Conversely, if $x \in$R, then $2x - 1 = a^m + u$, where $a^m \in$ m-reg(R) for a *fixed positive integer* $m \geq 1$ and $u \in U$(R), thus $2x = a^m + (u + 1), i$ implies that $x = \frac{a^m}{2} + \frac{(1+u)}{2}$. Now since $\frac{a^m}{2}(y + y)\frac{a^m}{2} = \frac{a^m ya^m}{4} + \frac{a^m ya^m}{4} = \frac{a^m}{2}$, it follows that $\frac{a^m}{2}$ is m-rnc element and $\frac{1+u}{2} \in N$(R).□

3 Some relations over m-rnc rings and other types of rings

This section discusses the relations among m-rnc rings and clean rings, local rings, and π-regular rings. ■

Preposition 3.1
If R≠ 0 is directly finite m-rnc ring and Id(R)={0,1}, so R is clean ring.
Proof
Since R is $m - rnc$ ring, then any $x \in$R has the form $x = a^m + b$, where $a^m \in m -$ reg(R) for a fixed positive integer $m \geq 1$ and $b \in N$(R). If $a^m = 0$, then $x = b = 0 + b$, *so* x is nil clean elementand hence $\mathcal{R}$ is nilclean ring. Therefore R is clean[3]. If $a^m \neq 0$, then there exists $y \in$R such that $a^m y\ a^m = a^m$. Thus $a^m y \in Id$(R) by hypothes is $a^m y = 0$ or $a^m y = 1$ *if* $a^m y = 0$, then $a^m = 0$ which is contradiction. Hence $a^m y = 1$, R is directly finite, so $y\ a^m = a^m y = 1$. Thus $a^m$is unite and x is clean. Therefore R is clean.■

Preposition 3.2
Every local ring is m-rnc ring.
Proof :
Suppose that R is local ring and $x \in$R thus either x is unit element or nilpotent element [10]. If x is unit, then $x \in m - reg$(R), then we can write $x = x + 0$, or if x is nilpotent element, thus $x = 0 + x$.■

Example:

Z_9, Z_8 is local ring and 3-regular nil clean ring . Now we give the necessary condition to prove every m-rnc ring is local.

Proposition 3.3

Assume R be m-rnc ring and $\{0,1\} \in$ Idm (R) with m-reg(R) ⊆ C(R). Then R is local ring and J (R) ⊆ N (R).

Proof:

Suppose R is m-rnc ring and $x \in$R, then $x = a^m + b$, where $a^m \in$ m-reg(R) with a positive integer $m \geq 1$ and $b \in N$(R), if $a^m = 0$, then $x = b$ is nilpotent element. If $a^m \in U$(R) , then $x = u + b$, which commute with other and $x \in U$(R) by (proposition 2.9) this mean that either x or $(1 - x)$ is unity element, hence R is local ring [12].

Now, to prove J (R)⊆ N (R) assume that $x \in J$(R) since R is local ring then x is either nilpotent or unit element[11], if x is unit, contradiction, since $x \in J$(R), hence$(1 - rx)$ is unite element in R for all $r \in$R, thus x is nilpotent element therefore J (R) ⊆ N (R). □

Proposition 3.4

Suppose R a ring. Then R is π-regular, if and only if , R is m-rnc ring, when R is NJ-ring.

Proof:

Let R is m-rnc ring and $x \in$R thus $x = a^m + b$,where $a^m \in$ m-reg(R) for a $fixed\ positive\ integer\ m \geq 1$ and $b \in N$(R), since R is NJ-ring, so $b \in J$(R)= 0 and hence $x = a^m = a^m\ r\ a^m\ is$ π-regular element. Therefore R is π-regular ring. ■

Example:

$(Z_4$, + , .) is m-rnc ring and π-regular.

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