m-rnc rings

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Abstract: In this article, an element w of an associative ring R is called m-regular nil clean or m-rnc if expressed as $w = a^m + b$ where $a^m$ is m-regular element and b is a nilpotent element. R is named m-regular nil clean ring or m-rnc ring. If all the elements of a ring R are m-rnc, some characteristics and basic properties of m-rnc rings are presented in this work.

1 Introduction

Following [1], [2], [3], [4] a ring R is called clean if all elements w in R it can be expressed as a sum of a unity $u \in R$ and an idempotent $e \in R$ similarly Diesl and Alexander [3] developed the nil clean ring, which told that R be nil clean when all elements in R can be expresses as a sum of an idempotent $e \in R$ and nilpotent $b \in R$. A ring R is considered to be m-regular ring if and only if there is an element $y \in R$ for each $a \in R$ and a positive integer $m \geq 1$ such that, $a^m = a^m y a^m$ [5], [6] when $m=1$, that is $a= a y a$, R is consider to be von Neumann. regular ring (or simply regular) [5], [6], [7] generally, all rings will be unitary and associative. For any ring, $N(R), J(R), R, Idem(R), m-reg (R), and C(R)$ are used to denote the set of all nilpotent elements, the group of unities element, Jacobson radical element of R, the right singular ideal of R, the set of the idempotent element in R, the set of m-regular elements of R and the center of R respectively. also, a ring R is denoted $N_f - ring$ if $N(R) \leq J(R)$ [8]. R named reduced when $N(R) = \{0\}, [9]$. Finally McCoy in [5] called a ring R to be $\pi$-regular, if $\forall a \in R, \exists$ positive integer $n$ and $b \in R$, so $a^n = a^nb^n$.

2 m-regular nil-clean rings

The definition of m-regular nil clean element and m-regular nil – clean rings and their characteristics also basic properties are presented in this section.

Definition 2.1

The element $x \in R$ is m-regular nil-clean or (m-rnc) if $x = a^m + b$ where $a^m \in m \text{ reg}(R)$ for a fixed positive integer $m \geq l$ and $b \in N(R)$. R is named m-regular nil-clean ring, or (m-rnc ring) if every element in R is m-rnc.

Clearly m-regular rings and nil clean rings are m-rnc rings. However, in general, m-rnc rings may not be nil clean, $Z_6$ is m-rnc but not nil clean since 2, 5 cannot be written as a sum of nilpotent and idempotent of $Z_6$, also $Z_8$ is m-rnc ring for $m = 3$, but not regular.

In the following some basic properties for m-rnc element are considered.

Preposition 2.2

The element $x$ in R is m- rnc if and only if $1 - x$ is m- rnc element.

Proof:

Assume $x$ is m-rnc element, then $x = a^m + b$, where $a^m$ is m-regular element for a fixed integer $m \geq 1$ and b is a nilpotent element in R then,

$1 - x = 1 - a^m - b = -a^m + (1 - b).$ Now, $-a^m = -(a^m b a^m)$

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\[ (-a^m)(-b)(-a^m) \text{ since } (-a^m) \text{ is } m - \text{rnc element. But } (1 - b^n) = 1 \]
\[ = (1 - b)(1 + b + b^2 + \ldots + b^{n-1}) \text{ thus } (1 - b) \text{ is unit, hence } (1 - b) \text{ is regular.} \]
Therefore \[ 1 - x \text{ is } m - \text{rnc element } \rightarrow (1 - b) \text{ is regular } \rightarrow m \text{ regular there fore } 1 - x \text{ is } m - \text{rnc.} \]
Conversely: Suppose that \( (1 - x) \text{ is } m - \text{rnc element, so } 1 - x = a^m + b, \text{ where} \]
\[ a^m \in m - \text{reg}(R) \text{ and } b \in N(R). \text{ Hence } x = a^m + (b - 1) \text{ and} \]
then \( x = -a^m + (b - 1). \text{ Therefore } x \text{ is } m - \text{rnc element.} \]

**Proposition 2.3**
Let \( R \) be an abelian ring, if \( x \in R \) is m-rnc element, then \( x^n \) is m-rnc element for some positive integer \( n \) if and only if \( x \) is m-rnc element.

**Proof:**
If \( x \) is m-rnc element. Thus \( x = a^m + b \), where \( a^m \) is m-rnc for a fixed positive integer, \( m \geq 1 \) and \( b \in N(R). \) Now, we prove \( x^n \) is m-rnc by mathematical induction;
\[ x^2 = (a^m + b)^2 = a^{2m} + 2a^mb + b^2 = a^{2m} + b(2a^m + b). \]
Assume that for \( n = k - 1 \), the assumption is true, that is
\[ x^{k-1} = (a^m + b)^{k-1} = \sum_{n=0}^{k-1} \binom{k}{n} a^m^n b^{k-n} \]
\[ = (a^m)^{k-1} + \frac{(k-1)}{1!} (a^m)^{k-2} b + \frac{(k-1)(k-2)}{2!} (a^m)^{k-3} b^2 + \ldots + b^{k-1} \]
We must prove the assumption is true when \( n = k \),
\[ x^n = x \cdot x^{n-1} \]
\[ = (a^m + b) \left[ \left( a^m \right)^{k-1} + \frac{k(k-1)}{2!} (a^m)^{k-2} b + \frac{k(k-1)(k-2)}{3!} (a^m)^{k-3} b^2 + \ldots + b^{k-1} \right] \]
\[ = (a^m)^k + b \left[ (a^m)^{k-1} + \frac{k(k-1)}{2!} (a^m)^{k-2} b + \frac{k(k-1)(k-2)}{3!} (a^m)^{k-3} b^2 + \ldots + b^{k-1} \right] \]
Suppose that \( l = b \)
\[ (a^m)^{k-1} + \frac{k(k-1)}{2!} (a^m)^{k-2} b + \frac{k(k-1)(k-2)}{3!} (a^m)^{k-3} b^2 + \ldots + b^{k-1} \]
\[ \in N(R) \text{ Therefore } x^n = (a^m)^n + l \]
Now, if \( x^n \) is m-rnc element, to prove \( x \) m-rnc element, then \( x^n = a^m + b \) so \( x^n - a^m = b. \) Now \( x^n - a^m = (x - a^m)(a^m + a^m x + a^m x^2 + \ldots + x^{n-1}) \in N(R). \text{ Hence } x - a^m \in N(R), \)
thus \( x - a^m = b, \) so \( x = a^m + b, \) therefore \( x \) is \( m - \text{rnc element.} \]

**Proposition 2.4**
Let \( R \) be an abelian ring with \( 2 \in N(R) \) and \( x \) is m-rnc element. Then \( x^2 - x \) it will be m-rnc element.

**Proof:**
If \( x \) be m-rnc element, there exists \( a^m \in m - \text{reg}(R) \) for fixed integer \( m \geq 1, \)
and \( b \in N(R), \) so that \( x = a^m + b. \) Now \( x^2 = (a^m + b)^2 = a^{2m} + 2a^mb + b^2. \)
Hence \( x^2 - x = (a^{2m} + 2a^mb + b^2) - (a^m + b) = (a^{2m} - a^m) + b(b - 1 + 2a^m) \)
\[ = a^m(a^m - 1) + b(b - 1 + 2a^m). \text{ There for } x^2 - x \text{ is } m - \text{rnc element.} \]

**Proposition 2.5**
Every homeomorphic surjective image of m-rnc ring is m-rnc ring.

**Proof:**
Let \( \Phi: R \rightarrow H \) is homomorphism function from m-rnc ring \( R \) onto the ring \( H, \) let \( x \in R \) and \( h \in H \) so \( \Phi(x) = h, \) thus we can write \( x = a^m + b \) for some \( a^m \in m - \text{reg}(R) \) and \( b \in N(R), \) then \( h = \Phi(x) = \Phi(a^m + b) = \Phi(a^m) + \Phi(b) \) since \( a^m \in m - \text{reg}(R), \) that is \( a^m = a^m y a^m \) for some \( y \in R \) and a fixed positive integer \( m \geq 1 \) and since, \( b \in N(R), \) so \( b^n = 0, \) where \( n \in \mathbb{Z}^+. \) Now \( h = \Phi(a^m y a^m) + (\Phi(b))^n = \Phi(a^m) \Phi(y) \Phi(a^m) + \Phi(b^n) \)
such that for a fixed positive integer m and b
\[ a + e^{n}b \] nilpotent. Proof:

Proposition 2.7

Let N a nil ideal of a R. So R is m-rnc if R/N is m-rnc ring.

Proof:-

Suppose R /N is m-rnc ring and x + N ∈ R/N, then x + N = (b + N) + (a^m + N) so, x + N = a^m + N, thus x - a^m ∈ N that is x - a^m = b^N, it follows that x = a^m + b^N Therefore R is m - rnc ring.

Proposition 2.8

Let R be m-rnc ring and x ∈ R, if xR not contain zero idempotent. Then x is a sum of right unit element and nilpotent element.

Proof

Assume that xR contains no non zero idempotent, select a^m ∈ m-reg(R) for a fixed positive integer, m ≥ 1 and b ∈ N(R) such that x - 1 = a^m + b. Then x = a^m + (b + 1), where (b + 1) is unit element and a^m = a^m y a^m for a fixed positive integer m ≥ 1 and for some y ∈ R. Since xR a^m = (a^m + b) y a^m, then x(1 - y a^m) = (b + 1)(1 - ya^m)(b + 1)^{-1} = 0. Hence, 1 - ya^m = 0 and then ya^m = 1, follows that(a^m) is unit. Therefore x is the sum of a right unity and a nilpotent.

proposition 2.9

If R be an abelian m-rnc ring. then e R e is m- rnc ring.

Proof:-

Let x ∈ e R e ⊆ R, then there exists a^m ∈ m - reg(R) for a fixed integer m ≥ 1 and b ∈ N(R), such that x = a^m + b now, exe = eax e + ebe, e ∈ Idem (R), since a^m ∈ m - reg(R), then exists c ∈ R such that, a^m = a^m c a^m and, e a^m e = e a^m c a^m e = e a c e a, thus a^m e is m - reg(R) element in e R e, also (eb e)^n = e^n b^n e^n = 0. For some n ∈ z^+ that is e b e is nilpotent element in e R e, thus x = (a^m e).c.(a^m e) + ebe and therefore, e R e is m - rnc ring.

Preposition 2.11

Suppose that R be m-rnc ring. Then J(R) is nilpotent ideal.

Proof:

\[ x ∈ J(R), since R is m - rnc ring, then x = a^m + b where a^m ∈ m - reg(R) \]
for a fixed integer m ≥ 1 and b ∈ N(R). Now, we must prove that a^m ∈ N(R),
\[ a^m = a^m c a^m, for some c ∈ R, it follows that (1 - a^m c)a^m = 0 \]
since \( x \in J(R) \), then \( a^m \in J(R) \) and \( a^m c \in J(R) \), hence \((1 - a^m c)\) is unit [8]. Then there exists \( u \in R \) such that 
\( u(1 - a^m c) = 1 \) multiply by \( a^m \) from the right, we get 
\( u(a^m - a^m c \ a^m) = a^m \),
that is \( a^m = 0 \). Therefore \( x \) is nilpotent element and \( J(R) \) is nilpotent ideal. ■

corollary 2.12
Assume \( R \) a reduced m-rcn ring. So \( J(R) = (0) \).

proof:
Let \( J(R) \neq (0) \), thus exists non-zero element \( x \in J(R) \), by proposition (2.11)
we have \( x \) is nilpotent element. Since \( R \) is reduced ring then \( x = 0 \). Hence \( J(R) = 0 \).

preposition 2.13:
If \( R \) is \( m - rnc \) ring, then \( y(R) \) is nilpotent ideal.

proof:
Suppose that \( x \in y(R) \), that \( x \) is \( m - rnc \) element, it follows that \( x = a^m + b \),
\( a^m \in m - reg(R) \) such that \( a^m = a^m c a^m \) for a fixed integer \( m \geq 1 \) and \( c \in R \) and \( b \in N(R) \).
Now to prove \( x \) is nilpotent, since \( x \in y(R) \), implies that \( a^m c \in Y(R) \), this mean that \( r(a^m c) \) is right singular ideal in the ring \( R \), let \( r(a^m c) \cap a^m R \neq 0 \),
then there exists \( 0 \neq y \in r(a^m b) \cap a^m R \) and we obtain \( a^m c y = 0 \) and \( y = a^m r \), that \( a^m c a^m r = a^m c y = 0 \),
where \( r \in R \), that is \( r(a^m c) \) is right singular ideal, thus \( a^m R = 0 \), thus \( a^m = 0 \). Hence \( x = 0 + b \) is nilpotent element and therefore \( y(R) \) is nilpotent ideal. ■

Proposition 2.14
Suppose that \( R \) be a ring so that \( m-reg(\mathcal{R}) \subseteq C(\mathcal{R}) \). If \( R \) is m-rcn ring. Then \( C(\mathcal{R}) \) is m-rcn ring.

proof:
Let \( R \) is m-rcn ring, and assume that \( x \in C(\mathcal{R}) \), then \( x \) can be write as; \( x = a^m + b \), where \( a^m \) is \( m - reg(\mathcal{R}) \) for a fixed positive integer \( m \geq 1 \) and \( b \in N(\mathcal{R}) \),
that is \( a^m = a^m y a^m \) by assume \( \ell = y a^m y \), that is \( \ell = y a^m y a^m \), multiply by \( a^m \), implies that \( a^m \ell a^m = a^m y a^m \), hence \( a^m \ell a^m = a^m y a^m = a^m \),
thus \( a^m \ell a^m \in m - reg(\mathcal{R}) \subseteq C(\mathcal{R}) \), which implies that \( b = (x - a^m) \in C(\mathcal{R}) \).

There fore \( C(\mathcal{R}) \) is \( m - rnc. \square \)

Proposition 2.15
For \( R \) abelian ring and \( P \) is the primitive ideal, and \( R/P \) be m-rcn ring. Then \( P \) is the maximal ideal in \( R \).

proof:
Let \( x \in R \), thus \( x + \rho \in R/P \), since \( R/P \) is m-rcn ring, so there exists
\( a^m \in m-reg(R) \) for a fixed positive integer \( m \geq 1 \) and \( b \in N(R) \) such that;
\[
\begin{align*}
x + \rho &= (a^m + b) + \rho = (a^m y a^m + b) + \rho, \text{ for some } y \in R, \text{ it follows that } \\
x + \rho &= (a^m + b) + (b + \rho) \\
&= (a^m + b) + (b + \rho) \\
&= (a^m y a^m + b)
\end{align*}
\]
Hence \( x - (a^m y a^m + b) \in \rho \), that is \( (a^m + b) - (a^m y a^m + b) \in \rho \), implies that, \( a^m - a^m y a^m \in \rho \) and thus \( a^m (1 - y a^m) \in \rho \), assume that \( a^m \in \rho \), then \( (1 - y a^m)^n \in \rho \)
for some positive integer \( n \).

Now \( (1 - y a^m)^n = 1 - \left( \sum_{k=1}^{n} c_n^k (1)^{n-k} y^k a^m (k-1) \right) a^m \in \rho \) where \( c_n^k = \frac{n!}{k!(n-k)!} \)
and let \( z = \sum_{k=1}^{n} c_n^k (1)^{n-k} y^k a^m (k-1) \), then \( 1 - 2a^m \in \rho \), it follows,
\[
1 + \rho = (z + \rho)(a^m + \rho) = (za^m + \rho)(a + \rho)
\]
hence \( a + \rho \) has inverse, thus \( R/P \) is the division ring, and so that \( P \) is the maximal ideal of \( R \). ■

Proposition 2.16
Let \( R \) a ring and \( r(x^{m+1}) \subseteq r(x^m) \), for all \( x \in R \) and positive integer \( m \geq 1 \), then \( R \) is m-rcn ring if \( R/r(a) \) is m-rcn ring.

proof:
Suppose that \( R/r(a) \) is m-rcn ring, thus \( x + r(a) \in E/R(a) \), where \( x \in R \) and \( x = a^m + b \) for \( a^m \in m - reg(R) \), for a fixed positive integer \( m \geq 1 \) and \( b \in N(R) \) so that
\[
\begin{align*}
x + r(a) &= (a^m + b) + (a) \\
&= (a^m + r(a) + (b + r(a))
\end{align*}
\]
\[
(a^m y a^m + r(a)) = (a^m y a^m + b) + r(a), \text{ which implies that } (a^m + b) + r(a) = (a^m y a^m + r(a)) + (b + r(a)) \text{ that is } (a^m + b) - (a^m y a^m) \in r(a) \text{ it follows that }
\]
\[
a^m - a^m y a^m \in r(a), \text{ that is } a (a^m - a^m y a^m) = 0 \text{ then } a^{m+1} (1 - ya^m) = 0
\]
Hence \((1 - y a^m) \in r (a^{m+1}) \subseteq r (a)\), implies that \(a^m(1 - y a^m) = 0\).

We obtain that, \(a^m = a^m y a^m\), therefore \(R\) is m-rnc ring.

**Proposition 2.17**

A direct product \( R = \prod_{i \in I} R_i \) of rings \( \{R_i\}_{i \in I} \) is m-rnc ring if and only if \( R_i \) is m-rnc ring for each \( i \).

**Proof:**

Let \( R \) is m-rnc ring, such as each of \( R_i \) is m-rnc ring by proposition 2.5 Conversely, assume that each of \( R_i \) for every \( i \) and set \( x = (x_i)_{i \in I} \in \prod_{i \in I} R_i \). For each \( i \), write \( x_i = a_i^m + b_i \) where \( a_i^m \in \text{m-reg}(R) \) for a fixed positive integer \( m \geq 1 \) and \( b_i \in N(R) \).

Since \( a_i^m \in \text{m-reg}(R) \), there exists \( y_i \in R_i \) such that \( a_i^m y_i a_i^m = a_i^m \), thus \( x = (a_i^m)_{i \in I} + (b_i)_{i \in I} \) Where \((a_i)_{i \in I} \in \text{m-reg}(\pi_{i \in I} R_i)\) and \((b_i)_{i \in I} \in N (\pi_{i \in I} R_i)\) Therefore \( \pi_{i \in I} R_i \) is m-rnc ring.

**Proposition 2.18**

Suppose \( R \) is a ring in which \( m \) is invertible. Then \( R \) is m-rnc, if and only if, every element of \( R \) is the sum of m-reg(\( R \)) and unite.

**Proof:**

Let \( R \) is m-rnc ring and \( x \) is any element of \( R \), thus \( \frac{x + 1}{2} \in R \), follows that \( \frac{x + 1}{2} = a^m + b \) where \( a^m \in \text{m-reg}(R) \) for a fixed positive integer \( m \geq 1 \) and \( b \in N(R) \), implies that

\[
x = 2a^m + (2b - 1), \text{ hence there exists } y \in R \text{, such that } a^m y a^m = a^m. \text{ Thus } (a^m + a^m) \frac{y}{2} (a^m + a^m) = \frac{a^m y a^m + a^m y a^m + a^m y a^m}{2} = \frac{1}{2}(a^m + a^m + a^m + a^m) = 2a^m.
\]

Thus \( 2a^m \) m-reg(R) and \( (2b - 1) \in U(R) \), such that \( x \) is a sum of is m-reg (\( R \)) element and a unite.

Conversely, if \( x \in R \), then \( 2x - 1 = a^m + u \), where \( a^m \in \text{m-reg}(R) \) for a fixed positive integer \( m \geq 1 \) and \( u \in U(R) \), thus \( 2x = a^m + (u + 1) \), implies that \( x = a^m + \frac{1 + u}{2} \). Now since \( a^m = a^m \frac{y + y}{2} = \frac{a^m y a^m + a^m y a^m}{4} = \frac{a^m}{2} \), it follows that \( \frac{a^m}{2} \) is m-rnc element and \( \frac{1 + u}{2} \in N(R) \)\( \Box \)

3 Some relations over m-rnc rings and other types of rings

This section discusses the relations among m-rnc rings and clean rings, local rings, and \( \pi \)-regular rings. ■

**Proposition 3.1**

If \( R \neq 0 \) is directly finite m-rnc ring and \( Id(R) = \{0, 1\} \), so \( R \) is clean ring.

**Proof**

Since \( R \) is m-rnc ring, then any \( x \in R \) has the form \( x = a^m + b \), where \( a^m \in m - \text{reg}(R) \) for a fixed positive integer \( m \geq 1 \) and \( b \in N(R) \). If \( a^m = 0 \), then \( x = b = 0 + b \), so \( x \) is nil clean elementand hence \( R \) is nilclean ring. Therefore \( R \) is clean.[3]. If \( a^m \neq 0 \), then there exists \( y \in R \) such that \( a^m y a^m = a^m \). Thus \( a^m y \in Id(R) \) by hypothesis is \( a^m y = 0 \) or \( a^m = 1 \) if \( a^m y = 0 \), then \( a^m = 0 \) which is contradiction. Hence \( a^m y = 1 \), \( R \) is directly finite, so \( y a^m = a^m y = 1 \). Thus \( a^m \) is unite and \( x \) is clean. Therefore \( R \) is clean. ■

**Proposition 3.2**

Every local ring is m-rnc ring.

**Proof:**

Suppose that \( R \) is local ring and \( x \in R \) thus either \( x \) is unit element or nilpotent element [10]. If \( x \) is unit, then \( x \in m - \text{reg}(R) \), then we can write \( x = x + 0 \), or if \( x \) is nilpotent element, then \( x = 0 + x \).

Example:
Z_5, Z_6 is local ring and 3-regular nil clean ring. Now we give the necessary condition to prove every m-rnc ring is local.

Proposition 3.3
Assume R be m-rnc ring and \{0,1\} ∈ \text{Idm}(R) with \text{m-reg}(R) \subseteq C(R). Then R is local ring and J(R) \subseteq N(R).

Proof:
Suppose R is m-rnc ring and \(x \in R\), then \(x = a^m + b\), where \(a^m \in \text{m-reg}(R)\) with a positive integer \(m \geq 1\) and \(b \in N(R)\), if \(a^m = 0\), then \(x = b\) is nilpotent element. If \(a^m \in U(R)\), then \(x = u + b\) which commute with other and \(x \in U(R)\) by (proposition 2.9) this mean that either \(x\) or \((1 - x)\) is unity element, hence R is local ring \[12\].

Now, to prove \(J(R) \subseteq N(R)\) assume that \(x \in J(R)\) since R is local ring then \(x\) is either nilpotent or unit element\[11\], if \(x\) is unit, contradiction, since \(x \in J(R)\), hence \((1 - rx)\) is unite element in R for all \(r \in R\), thus \(x\) is nilpotent element therefore \(J(R) \subseteq N(R)\). □

Proposition 3.4
Suppose R a ring. Then R is \(\pi\)-regular, if and only if, R is m-rnc ring, when R is NJ-ring.

Proof:
Let R is m-rnc ring and \(x \in R\) thus \(x = a^m + b\), where \(a^m \in \text{m-reg}(R)\) for a fixed positive integer \(m \geq 1\) and \(b \in N(R)\), since R is NJ-ring, so \(b \in J(R)\) = 0 and hence \(x = a^m\) is \(\pi\)-regular element. Therefore R is \(\pi\)-regular ring. ■

Example:
\(\mathbb{Z}_4, +, \cdot\) is m-rnc ring and \(\pi\)-regular.

References

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