# Intelligent Particle Swarm Optimization Method for Parameter Selecting in Regularization Method for Integral Equation

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**Abstract.** We use the Tikhonov method as a regularization technique for solving the integral equation of the first kind with noisy and noise-free data. Following that, we go over how to choose the Tikhonov regularization parameter by implementing the Intelligent Piratical Swarm Optimization (IPOS) technique. The effectiveness of combining these two approaches IPOS and Tikhonov is demonstrated to be highly practicable.

## 1 Introduction

The discretization method has been widely employed in numerous papers as a key approach for solving integral equations [1–4]. Scholars have estimated the fault of the estimated solution, considering the discretization of integral equations, in previous works [5-8]. In particular, when the integral equation of the first kind, there are many articles that debate and apply the regularization methods for solving integral equation problems [9]–[13] [14] [15], [16]. The success of these regularization techniques depends on carefully examining the mathematical inverse issues linked to the assertions of the works contents and categorizing obvious issues in their resolution [17-22].

There are significant search methods that incorporate population-based optimization. These methods belong to a group of algorithms that draw inspiration from nature, such as “PSO”. The “PSO” established by Eberhart and Kennedy [23] can locate global optimum solutions or accurate approximations for many types of inverse problems. In [24] the “PSO” used to determination electromagnetic inverse issue by presenting a new appropriateness excellent procedure, a dynamic parameter informing plan, and matching the inspection and manipulation quests. This methodology also contains a new place updating formulation to evade getting trapped in limited optima. In the paper [25] particle swarm optimization is employed to address inverse design issues for cylindrical thermal multilayer shielding and cloaking shells. These inverse issues are transformed into related control problems. The PSO technique is also utilized in [26-28].

To stabilize the result of the problem integral equation first kind, the method regularization Tikhonov is hired. The key to the success of Tikhonov lies in selecting the

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appropriate regularization parameter. This parameter controls the balance between fitting the data and achieving a smooth solution. IPSO is a nature-inspired optimization algorithm that has been successfully used in various fields to find the global minimum of a cost function. It is hypothesized that IPSO can effectively select the optimal parameter in the method of Tikhonov. By combining the Tikhonov with IPSO methods for parameter selection, a more stable and accurate solution to problems can be obtained. This improves the practicality of solving real-world problems. This paper explores the effectiveness of this combined approach through thorough mathematical analysis and numerical experiments on various case studies. Each step of this work is explained. Section 2 describes the problem and provides a mathematical analysis of the integral equation problem and the Tikhonov approach. Section 3 defines the IPSO algorithm and explains the crucial steps or sections for programming. Finally, the outcomes demonstrate the capability of IPSO to select a regularization parameter while disregarding noise.

*Problem Statement*

The next integral problem takes into consideration.

$$Iw(x) = \int_a^b \hat{K}(x,t)w(x)dx = h(t), c \le t \le d, \quad (1)$$

where $w(x) \in L_2[a, b]$, $h(t) \in L_2[c,d]$, the $\hat{K}(x,t)$signify the kernel operator $I$, where$\hat{K}(x,t), \hat{K}_t'(x,t) \in C^{1,1}([a,b] \times [c,d])$. We propose for $h(t) = h_0(t)$ there exist a true solution $w_0(x)$ for problem (1) in the set $H_r$

$$H_r = \{w(x)\colon\ w(x), w'(x) \in L_2[a,b], w(a) = w(b) = 0, \|w(x)\|_{L_2}^2 \le r^2\}, \quad (2)$$

The function $h_0(t)$ is unidentified in its place we have $h_\delta \in L_2[c,d]$ and $\delta > 0$ such that $\|h_\delta(t) - h_0(t)\|_{L_2}^2 \le \delta^2$. For resolving the (1) we need discovery the solution $w_\delta(x)$ by using $h_\delta(t)$, $\delta$, and $H_r$. Then estimated the deviation of the calculation solution $w_\delta(x)$ from the $w_0(x)$ in $L_2[a,b]$, $\|w_\delta(x) - w_0(x)\|_{L_2}$

Let describe operator $Q\colon\ L_2[a,\ b]\ \to L_2[a,\ b]$ by

$$w(x) = Qv(x) = \int_a^s v(\varphi)d\varphi\,,\ v(x), Qv(x) \in L_2[a,\ b], \quad (3)$$

There is an operator named $C$ which can be defined by

$$Tv(x) = IQv(x),\ \ v(x), \in L_2[a,\ b],\ \ Tv(x) \in L_2[c,\ d], \quad (4)$$

by (3) and (4) we get $Tv(x) = \int_a^b P(x,t)\, v(x)dx$, where $P(x,t) = -\int_a^b \hat{K}(\eta,t)\, d\eta$.

The operator $T_{n,\,m}$ has finite-dimensional property and it has been defined in order to calculating the mathematical for (1), $T$ swapped with the $T_{n,\,m}$. Next step it is necessary split intervals $[a,b]$ and $[c,d]$ into $n$ and $m$ equivalent slices respectively. Where interval $[a,b]$ divided $x_i = a + \frac{i(b-a)}{n}, i = 0, 1, \dots, n-1$, and interval $[c,d]$ divided by points $t_j = c + \frac{j(d-c)}{m}, j = 0, 1\ \dots, m-1$. Now define next

$$P_i(t) = P(x_i, t), \quad (5)$$

$$P_n(x,t) = P_i(t); x_i \le x \le x_{i+1},\quad t \in [c,d],\ i = 0,\ 1,\ \dots,\ n-1 \quad (6)$$

$$P_{n,m}(x,t) = P_i(t_j); \quad (7)$$

where $x_i \le x \le x_{i+1}$, $t_j \le t \le t_{j+1}$, $i = 0,\ 1,\ \dots,\ n-1, j = 0,\ 1,\ \dots,\ m-1$, By (5)–(7) we get $P_{n,}$ and $P_{n,\,m}$

$$T_n v(x) = \int_a^b P_n(x,t)\, v(x)dx, \text{ and } T_{n,m}v(x) = \int_a^b P_{n,m}(x,t)\, v(x)dx, \quad (8)$$

The (1) reduce to linear operator problem.

$$T_{n,m}v(x) = h_\delta(t), \quad (9)$$

The estimate solution for (1) can be found by consuming the general "discrepancy method" proposed in [29] and studied in [30]. The method reduces the (9) to the "conditional extremum variational" problem.

$$inf\{\|v(x)\|^2, \|T_{n,m}v(x) - h_\delta(t)\| \leq \|v(x)\| + \delta\}, \quad (10)$$

For resolving (10) we get

$$T_{n,m}^T T_{n,m} v(x) + \alpha v(x) = T_{n,m}^T h_\delta(t). \quad (11)$$

for rewriting the (11) in the form of vector and matrix

$$(T_{j,i})^T T_{j,i} \bar{v}_i + \alpha \bar{v}_i = (T_{j,i})^T T_j,$$

where $i = 0,1,\ldots n-1, j = 0,1,\ldots,m-1$, and

$$\bar{v}_i == \begin{bmatrix} \bar{v}_0 = \int_{x_0}^{x_1} v(x)dx \\ \bar{v}_1 = \int_{x_1}^{x_2} v(x)dx \\ \vdots \\ \bar{v}_{n-1} = \int_{x_{n-1}}^{x_n} v(x)dx \end{bmatrix}$$

$$h_j = \begin{bmatrix} h_0 = \int_{t_0}^{t_1} h(t)dt \\ h_1 = \int_{t_1}^{t_2} h(t)dt \\ \vdots \\ h_{m-1} = \int_{t_{m-1}}^{t_m} h(t)dt \end{bmatrix}$$

$$T_{j,i} = \begin{bmatrix} T_{i=0}(t_{j=0}) & T_{i=1}(t_{j=0}) & \cdots & T_{i=n-1}(t_{j=0}) \\ T_{i=0}(t_{j=1}) & T_{i=1}(t_{j=1}) & \cdots & T_{i=n-1}(t_{j=1}) \\ \vdots & \vdots & \ddots & \vdots \\ T_{i=0}(t_{j=m-1}) & T_{i=1}(t_{j=m-1}) & \cdots & T_{i=n-1}(t_{j=m-1}) \end{bmatrix}$$

$$(T_{j,i})^T = \begin{bmatrix} T_{i=0}(t_{j=0}) & T_{i=0}(t_{j=1}) & \cdots & T_{i=0}(t_{j=m-1}) \\ T_{i=1}(t_{j=0}) & T_{i=1}(t_{j=1}) & \cdots & T_{i=1}(t_{j=m-1}) \\ \vdots & \vdots & \ddots & \vdots \\ T_{i=n-1}(t_{j=0}) & T_{i=n-1}(t_{j=1}) & \cdots & T_{i=n-1}(t_{j=m-1}) \end{bmatrix}$$

## 2 PSO for parameter α selecting

In the following we consider IPOS way for selecting the Tikhonov regularization parameter α, without need known the noise level δ.

The fundamental concept of IPSO can be stated clearly as follows [33]. Each member of the swarm, known as a "particle," represents a possible solution; every particle modifies its position in the search domain and updates its velocity in accordance at each iteration with its own and its "neighbors'" flying experiences, striving for an improved position for itself provided that it meets certain fitness requirements.

The “current best position” (*cbest*), which is the best engaged site among all affiliates from the start of the search until end, it is efficient and learned iteratively by the algorithm. Additionally, each particle memorizes its personal best position (*pbest*), also known as its best experienced position. The equation $\Delta x = e\Delta t$, where *e* characterizes the velocity, is used to compute the distance each particle travels to reach its subsequent position. It is calculated for particle *i* at iteration *t* as follows:

$$e_i^{t+i} = \omega e_i^t + c_1 r_1 d_c^{i,t} + c_2 r_2 d_p^{i,t} \quad (12)$$
$$d_p^{i,t} = [x_i^t - pbest^{i,t}] \quad (13)$$
$$d_c^{i,t} = [x_i^t - cbest^t] \quad (14)$$

The distances between a particle's current position and its best individual and global positions are $d_c^{i,t}$ and $d_p^{i,t}$, respectively. The ω is defined as inertia weight, $r_1$ & $r_2$ are uniform random numbers, and $c_1$ & $c_2$are called acceleration coefficients.
$\Delta t$ is equal to 1 between any two subsequent iterations (i.e.,$\Delta t = (t+1) - t = 1$), hence particle *i* will be in the following position at iteration $(t+1)$:

$$a_i^{t+1} = a_i^t + v_i^{t+1} \quad (15)$$

Now, the IPSO calculation methods are applied by the steps that follow to solve the problem.
step 1: Define the problem and the IPSO.
In this part, we define integral equation of the first kind and the IPSO. We create the fitness or "cost function" *(CostFunction)*. Then, we have to set up the subsequent variables and function:

- Set domain size *n*
- Define cost function:

*CostFunction (v, A, f, n)*
*{*

$$I = eye(n, n)$$
$$x = inv(T^*T + \alpha I) * T^*h$$

*return* $\|Tv - h_\delta\|^2 + \quad \|\alpha\|\|x\|^2$
*}*

- Set *nVar; / "*number of unknown decisions variables"
- Set *VarSize = [1, nVar]; / "*matrix size of decision variables"
- Set VarMin *;* /"lower bound of decision variables"
- Set VarMax; / "upper bound of decision variables"

Step 2: Parameters.

- "Set maximum of iterations" *MaxIt*
- "Set population size" (swarm size) *nPop*
- Set *w*
- Set $c_1$
- Set $c_2$

Step 4: Initialize population members

- "Define the arrays for particle": *Position = [ ]; Velocity = [ ]; Cost = [ ]; Best.Position = [ ]; Best.Cost = [ ];*
- "Define the current best" *Current Best = inf;*
- "Loop under population size" (*nPop*)
- "Create random" *Position = unifrnd(VarMin, VarMax, VarSize);*
- "Adjust velocity" *Velocity = zeros (VarSize);*
- "Valuation " *Cost = CostFunction(Position);*
- "Update best position and cost by" *Best.Position and Best.Cost = Cost;*
- "Update the current best (position and cost) by "
  If *Best.Cost < Current.Best Cost*
  *Current.Best = Best.Position and Best.Cost = Cost*
- End loop over *nPop*

Step 5: Main loop of PSO

- Loop over iterations *it=1: MaxIt*
  - Loop under population size (*nPop*)
    - Update velocity:

*Velocity = w * Velocity + c1*rand(VarSize) .* Best.Position - Position+ c2*rand(VarSize) .* Current.Best.Position - Position);*

- Update position"
  *Position = Position + Velocity;*
- Evaluation:
  *Cost = CostFunction(Position);*
- Update
  *if Cost < Best.Cost*
  *Best.Position = Position;*
  *Best.Cost = Cost;*
  - Update
    *if Best.Cost < Current.Best.Cost*
    *Current.Best = Best;*
    *End*

  *End*

- End loop over *nPop*
- Store the best cost value *Best.Costs(it) = Current.Best.Cost;*
- End loop over iterations *MaxIt*

Step 6: Obtain the results.

- Set *α= Best.Costs(it);*
- Find approximation solution for inverse problem by: $v = inv(C^*C + \alpha I) * C^*f,$

## 3. Numerical experiments

In this section we study the Fredholm integral equation of the first kind as the generic for our testing

$$\int_a^b P(s,t)w(t) = h(t), \quad c \le s \le d, \qquad (16)$$

After discretizing (16), we get

$$min\|Tw - h\|_{L_2} \quad (17)$$

The Fredholm integral equation(16) is discretized to produce each example. The discrete ill-posed issue (17) is the discretized form. The efficiency of the PSO algorithm for parameter selection in the Tikhonov method is examined in this section. We give various instances that demonstrate how the PSO performs and support its accuracy and effectiveness. We consider noise-free and noisy data without used in parameter selection process.

Example 1 (Phillips test) The MATLAB function Phillips from [34] is used to discretize the integral equation using the Galerkin approach. The vectors f, u, and matrix A are the results. A is an indefinite, symmetric coefficient matrix. The matrix's singular values steadily decrease to zero. The matrix is in ill-condition case.

Example 2 (Gravity test) To create the matrix *A*, a symmetric Toeplitz matrix, we employ the gravity function from [34]. The depth of the magnetic deposit is a parameter for the test called d. In our testing, we use the parameter's default value of *d = 0.25* However, selecting a higher number for d causes the single values to decay more quickly. The matrix *A* and vector *u* are produced using the Matlab function gravity. The right side is then calculated using the formula *f = Au*.

The values for all PSO algorithm parameters with measured and unmeasured noise levels are shown in Tables 1 and 2. The all tests with nose-free and nose level have the commune parameters value in PSO as following:

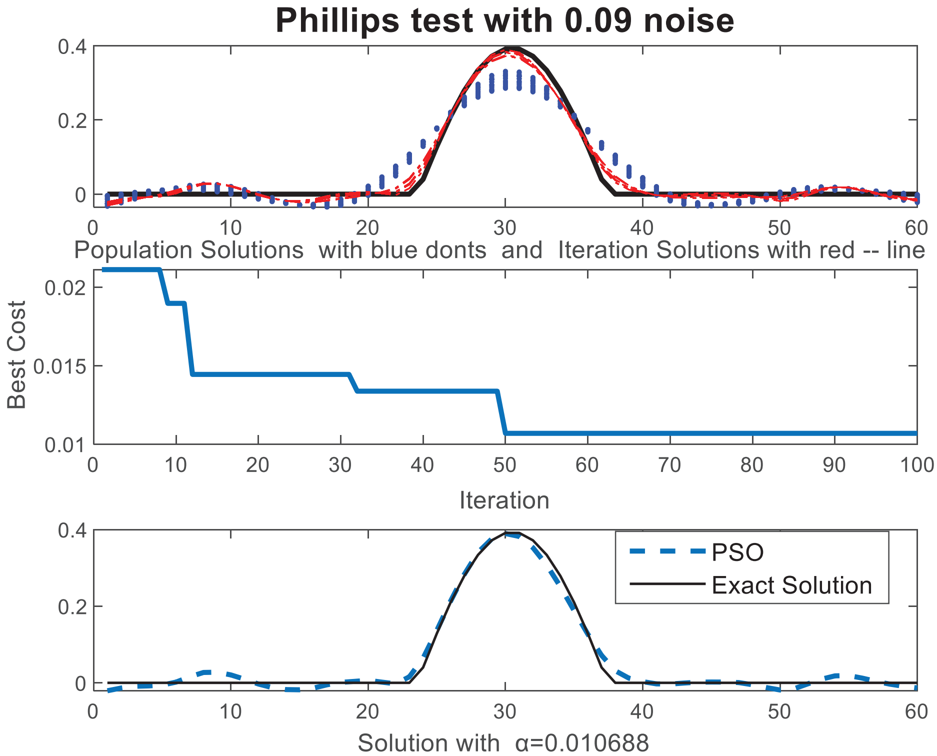
- *nVar = 1*
- *VarSize = [1, nVar]*

- *VarMin = 0;*
- *VarMax = 1;*
- *c1 = c2 = 2.*

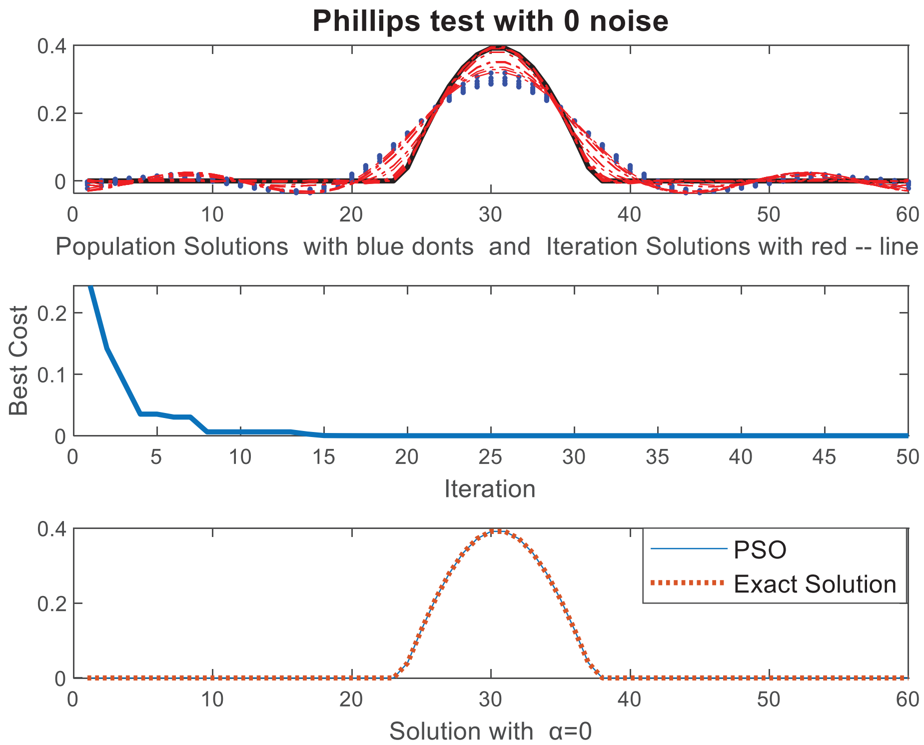
**Table 1.** PSO parameters with unknown noise level

| Name of Parameter | Philips test | | Gravity test | |
|---|---|---|---|---|
| | nose-free | with nose | nose-free | with nose |
| MaxIt | 50 | 100 | 30 | 50 |
| nPop | 5 | 20 | 4 | 10 |

In our three experiments, POS techniques produce nearly satisfactory outcomes for a range of noise levels as well as for noise-free situations. We present the findings for the current best regularization parameter in Figures. 1, and 2 along with the results for the precise solution compared with population selection in the initial portion and solutions in the iteration section.

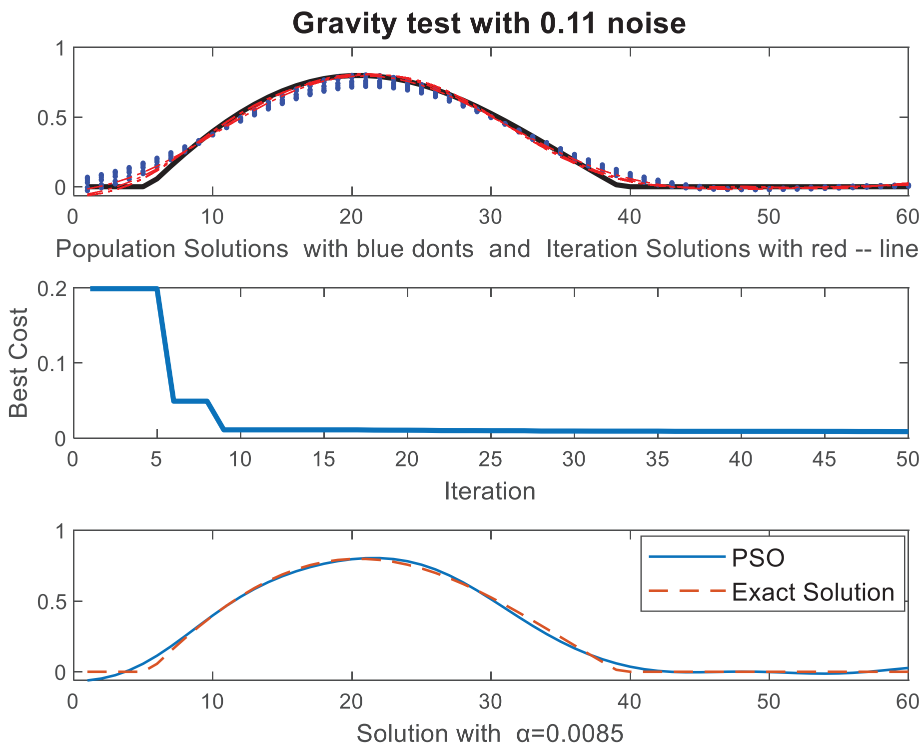



(a)

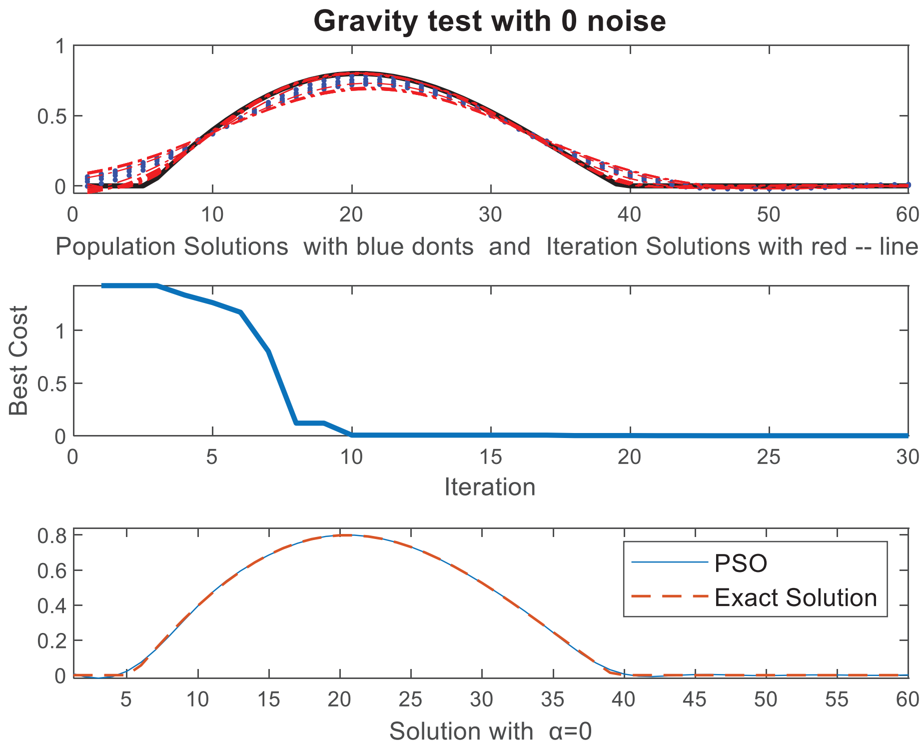



(b)

**Fig. 1**. Phillips test




(a)




(b)

**Fig. 2.** Gravity test

## 4 Conclusion

In this article, we review the regularization Tikhonov approach and IPSO for resolving the integral equation with noisy and noise-free data. We offer a mathematical analysis of a disretization step for integral equation. To ensure that the approximation to the exact solution converges, the regularization parameter for the Tikhonov regularization method must be chosen by new technique. IPSO worked as searcher for the best regularization parameter. Using the variational equation for the Tikhonov method as a cost function, that resulted in the least value for the variational equation. In numerical experiments various problems are employed as case studies such as the Phillips, and Gravity test. The numerical outcomes demonstrate the effectiveness of combining the Tikhonov regularization method with IPSO.

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