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On some types of nano contra β_{pc}-continuous functions

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Abstract: The purpose of this paper is to define and examine two novel forms of Nano β_{PC}- continuity, namely Nano contra β_{PC}- continuity and Nano almost contra β_{PC}- continuity, and to provide several fundamental characteristics and attributes related to these mappings. Moreover, , paper is done on Nano almost continuously. The connections between these mappings and other well-known functions are found.

1 Introduction

Using equivalency relations and approximations to define the boundary region of a subset of the universe, Lellis Thivagar et al [5] proposed the concept of Nano topology, where each element is referred to as a Nano open set. As a result, various concepts like "Nano closed," "Nano closure," and "Nano interior" are constructed. Also, the relationships between the boundary region, upper approximation, and lower approximation are examined. Numerous researchers have since examined the subject of Nano topology; see[3], [5]. A. Ravathy and Ganambal Ilgano[1] defined "Nano β -open" and identified the family of Nano β -open sets based on approximation difference scenarios.

Eventually, a new class of Nano continuity called Nano continuous function was established by Lellis Thivagar and Carmel Richard[7], who also determined how to characterize it in terms of Nano closure and Nano interior. The relationships between these functions are addressed. In 2019 P.K.Dhanasekaran et al.[9] introduced several assertions that meet the conditions of Almost Nano pre-continuous, Almost Nano semi-continuous, and Almost Nano alpha continuous functions.

A powerful variation of Nano continuity called as Nano contra continuity was created by Lellis Thivagar et al.[8] Additionally, the installation of economic and medical examples of Bi- continuity was studied.

2 Preliminaries

Definition 2.1 [6] considering $\mathcal{V}$ be universe, comprise of limited many things and RL_{eq} be an equivalence relation on RL_{eq} recognize as the indiscernibility relation. consequently $\mathcal{V}$ is partitioned to equivalence classes which are disjoint indistinguishable elements are those that

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belong to a certain equivalence class. A pairing $(\mathcal{V}, RL_{eq})$ is named approximation space .Letting $D \subseteq \mathcal{V}$

G1- The whole assortment of items with labels D with respect to $\mathfrak{R}$ is the lower approximation of X w.r.t. RL_{eq} ,which is symbolized by $L_{RL_{eq}}(D)$. that is $L_{RL_{eq}}(D) = \bigcup_{x\in\mathcal{V}}\{RL_{eq}(x): RL_{eq}(x) \subseteq D\}$ such that $RL_{eq}(x)$ is represented the set of the equivalence classes determined by x.

G2- The whole assortment of items with possibly tags as D regard to $\mathfrak{R}_q$ is namely upper approximation of X regarding to R_q. Equivalently, $U_{RL_{eq}}(D) = \bigcup_{x\in\mathcal{V}}\{RL_{eq}(x): RL_{eq}(x)\cap D \neq \phi\}$

G3- The set of any things can be categorized not D nor as non X with respect to RL_{eq} is known as the boundary region of X regards to R_q. Equivalently, $B_{RL_{eq}}(D) = U_{RL_{eq}}(D) - L_{RL_{eq}}(D)$.

Proposition 2.2.[6] If $(\mathcal{V}, RL_{eq})$ is an approximation space and $X, Y \subseteq \mathcal{V}$, then

(i) $L_{RL_{eq}}(X) \subseteq X \subseteq U_{RL_{eq}}(X)$.

(ii) $L_{RL_{eq}}(\phi) = U_{RL_{eq}}(\phi) = \phi$ and $L_{RL_{eq}}(\mathcal{V}) = U_{RL_{eq}}(\mathcal{V}) = \mathcal{V}$.

(iii) $U_{RL_{eq}}(X\cup Y) = U_{RL_{eq}}(X)\cup U_{RL_{eq}}(Y)$.

(iv) $U_{RL_{eq}}(X\cap Y) \subseteq U_{RL_{eq}}(X)\cap U_{RL_{eq}}(Y)$.

(v) $L_{RL_{eq}}(X\cup Y) \supseteq L_{RL_{eq}}(X)\cup L_{RL_{eq}}(Y)$.

(vi) $L_{RL_{eq}}(X\cap Y) = L_{RL_{eq}}(X)\cap L_{RL_{eq}}(Y)$.

(vii) $L_{RL_{eq}}(X) \subseteq L_{RL_{eq}}(Y)$ and $U_{RL_{eq}}(X) \subseteq U_{RL_{eq}}(Y)$ whenever $X \subseteq Y$.

(viii) $U_{RL_{eq}}(X^c) = [L_{RL_{eq}}(X)]^c$ and $L_{RL_{eq}}(X^c) = [U_{RL_{eq}}(X)]^c$.

(ix) $U_{RL_{eq}}U_{RL_{eq}}(X) = L_{RL_{eq}}U_{RL_{eq}}(X) = U_{RL_{eq}}(X)$.

(x) $L_{RL_{eq}}L_{RL_{eq}}(X) = U_{RL_{eq}}L_{RL_{eq}}(X) = L_{RL_{eq}}(X)$.

Definition 2.3[6] Letting $\mathcal{V}$ stand for the universe and R_q stand for equivalence relation on $\mathcal{V}$ equipped with $\tau_{R_q}(D) = \left\{\phi, \mathcal{V}, U_{RL_{eq}}(D), B_{RL_{eq}}(D), L_{RL_{eq}}(D)\right\}$ that is D is subset of $\mathcal{V}$ and $\tau_{RL_{eq}}(D)$ addressed the below conditions

T1- $\mathcal{V}$ and ϕ are elements of $\tau_{RL_{eq}}(D)$

T2- the union of all elements of subfamily of $\tau_{RL_{eq}}(D)$ is also, in $\tau_{RL_{eq}}(D)$.

T3- the intersection of finitely many subfamily of $\tau_{RL_{eq}}(D)$ is also, in $RL_{eq}(D)$.

Then the family $\tau_{RL_{eq}}(D)$ is known as Nano topology on $\mathcal{V}$ regarding to D and the pair $\left(\mathcal{V}, \tau_{RL_{eq}}(D)\right)$ is named a Nano topological space $\mathfrak{N}$.Top. S is our abbreviation.

The members of $\tau_{R_q}(D)$ are named as Nano open set. Also, the Nano open set's complement knows as Nano closed set. Nano open and Nano closed sets are abbreviated as N-open and N-closed respectively.

Definition 2.4[6] For a $\mathfrak{N}$.Top. S $\left(\mathcal{V}, \tau_{RL_{eq}}(X)\right)$ regarding to D where $D \subseteq \mathcal{V}$. If L is a subset of $\mathcal{V}$, the Nano interior of L is the union of every Nano open sets contained in L as well as its symbolized by $\mathfrak{N}int(L)$ and Nano closure of L is the intersection of all Nano closed sets containing L as well as its symbolized by $\mathfrak{N}clo(L)$.

Remark 2.5 [6] The collection $\beta = \left\{\mathcal{M}, L_{RL_{eq}}(D), B_{RL_{eq}}(D)\right\}$ forms a base for a Nano topology $\tau_{RL_{eq}}$.

Definition 2.6.[6] A subset H of $\mathfrak{N}$.Top. S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ is named

S1- Nano semi-open if $H \subseteq \mathfrak{N}clo\ \mathfrak{N}int(H)$
S2- Nano pre-open if $H \subseteq \mathfrak{N}int\ \mathfrak{N}clo(H)$
S3- Nano β-open if $H \subseteq \mathfrak{N}clo\ \mathfrak{N}int\ \mathfrak{N}clo(H)$
S4- Nano regular-open if $H = \mathfrak{N}int\ \mathfrak{N}clo(H)$.
The complements of the above mentioned sets are called their respective closed sets.

Proposition 2.7.[4] A $\mathfrak{N}$.Top. S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ is said to be Nano regular space if and only if for each $p \in \mathcal{M}$, and for each $K \in \mathfrak{N}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ containing p, there exists $H \in \mathfrak{N}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ such that $p \in H \subseteq \mathfrak{N}cl(H) \subseteq K$

Definition 2.8. **[12]** A $\mathfrak{N}$.Top. S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ is said to be Nano Pre $Frechet -$ space if for every two distinct points s, t there are two Nano Pre– open sets K and L contains one but not the another .

Proposition 2.9.[14]. If a $\mathfrak{N}$.Top .S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ is Nano pre $Frechet -$ space, then $\mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) = \mathfrak{N}\beta o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$.

Definition 2.10.[14] A Nano β-open subset B of $\mathfrak{N}$.Top.S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ is named Nano β_{PC}-open if for each $x \in B$, there is Nano pre-closed F such that $x \in F \sqsubseteq B$. the family of all β_{PC}-open sets is symbolled by $\mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$.

Proposition 2.11[14]: A subset K of $\mathfrak{N}$.Top.S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ is Nano β_{PC}-open set if and only if for any $p \in K$, there is $L \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ such that $p \in L \subseteq K$

Definition 2.12.[14] A Nano subset H of $\mathfrak{N}$.Top. S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ is named β_{PC}-closed if it`s complement is Nano β_{PC}-open. the family of all β_{PC}-closed sets is symbolled by $\mathfrak{N}\beta_{PC}c\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$.

Definition 2.13.[14] A point $p \in M$ is known as $\mathfrak{N}\beta_{PC}$ – interior point of $K \subseteq M$, if there is a $\mathfrak{N}\beta_{PC}$-open L ($L \in \mathfrak{N}P_{Pc}o(M, \tau_R(X))$ having p such that $p \in L$. the set of all $\mathfrak{N}\beta_{PC}$ -interior point of K is named $\mathfrak{N}\beta_{PC}$ -interior and symbolled by $\mathfrak{N}P_{Pc}int(K)$.

Theorem 2.14.[14] Let K be a subset of $\mathfrak{N}$.Top. S. $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$. Then

D1) $\mathfrak{N}\beta_{PC}int(K) \subseteq K$
D2) $K \in \mathfrak{N}\beta_{PC}o(M, \tau_R(X))$ if and only if $K = \mathfrak{N}\beta_{PC}int(K)$
D3) $\mathfrak{N}\beta_{PC}int(\phi) = \phi\ and\ \mathfrak{N}\beta_{PC}int(M) = M$
D4) if $A \subseteq B\ ,\ then\ \ \mathfrak{N}\beta_{PC}int(A) \subseteq \mathfrak{N}\beta_{PC}int(B)$
D5) $\mathfrak{N}\beta_{PC}int(A) \cup \mathfrak{N}\beta_{PC}int(B) \subseteq \mathfrak{N}\beta_{PC}int(A \cup B)$
D6) $\mathfrak{N}\beta_{PC}int(A \cap B) \subseteq \mathfrak{N}\beta_{PC} - int(A)\ \cap\ \mathfrak{N}\beta_{PC} - int(B)$.

Definition 2.15.[14] Let A be a subset of $\mathfrak{N}$.Top. S. $(\mathcal{M}, \tau_R(X))$ is said to be $\mathfrak{N}\beta_{PC}$ -closure of A define as the intersection of all $\mathfrak{N}\beta_{PC}$-closed containing A , that means $\mathfrak{N}\beta_{PC}clo(A) = \cap \left\{ K \supseteq A : K \in \mathfrak{N}\beta_{PC}c(\mathcal{M}, \tau_R(X)) \right\}$.

Proposition2.16.[14] For any subset A and B of $\mathfrak{N}$.Top.S. $(\mathcal{M}, \tau_R(X))$, then
W1) If $A \subseteq B\ , then\ \mathfrak{N}\beta_{PC}clo(A) \subseteq \mathfrak{N}\beta_{PC}clo(B)$
W2) $\mathfrak{N}\beta_{PC}clo(A \cap B) \subseteq \mathfrak{N}\beta_{PC}clo(A) \cap \mathfrak{N}\beta_{PC}clo(B)$
W3) $\mathfrak{N}\beta_{PC}clo(A) \cup \mathfrak{N}\beta_{PC}clo(B) \subseteq \mathfrak{N}\beta_{PC}clo(A \cup B)$
W4) $K \subseteq \mathfrak{N}\beta_{PC}clo(K)$

W5) $\mathfrak{N}\beta_{PC}clo(\phi) = \phi$ and $\mathfrak{N}\beta_{PC}clo(M) = M$.

Definition2.17[13]. For a Nano subset H of $\mathfrak{N}$.Top.S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ is Nano θ-open(resp. , Nano semiθ-open , Nano preθ-open) set if for each $x \in H$, there exists a Nano open(resp. , Nano semi-open , Nano pre-open) set G such that $x \in G \subseteq \mathfrak{N}clo(G) \subset H$.

Definition 2.18.[7] Let $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ and $\left(\mathcal{H}, \tau^*_{RL_{eq}}(Y)\right)$ be two $\mathfrak{N}$.Top. S , a function $f: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{H}, \tau^*_{RL_{eq}}(Y)\right)$ is called Nano continuous if the inverse image of every Nano – open set in $\mathcal{H}$ is Nano open in $\mathcal{M}$.

Definition 2.19[8] Let $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ and $\left(\mathcal{H}, \tau^*_{RL_{eq}}(Y)\right)$ be two $\mathfrak{N}$.Top. S , a function $f: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{H}, \tau^*_{RL_{eq}}(Y)\right)$ is said to be Nano contra continuous, if the inverse image of every Nano open set in $\mathcal{H}$ is Nano closed in $\mathcal{M}$.

Definition 2.20[2] Let $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ and $\left(\mathcal{H}, \tau^*_{RL_{eq}}(Y)\right)$ be two $\mathfrak{N}$.Top. S , a function $f: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{H}, \tau^*_{RL_{eq}}(Y)\right)$ is called Nano β-continuous if the inverse image of every Nano – open set in $\mathcal{H}$ is Nano β -open in $\mathcal{M}$.

Definition 2.21.[14] A function $f: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$ is known as Nano β_{PC} -continuous ,if $f^{-1}(K)$ in M is Nano β_{PC} -open for every Nano open set K in $\mathfrak{N}$.

Definition 2.22.[14] Let $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ and $\left(\mathcal{H}, \tau_{RL^*{}_{eq}}(X)\right)$ be two $\mathfrak{N}$.Top. S, a function $f: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{H}, \tau_{RL^*{}_{eq}}(X)\right)$ is named β_{PC}-Irresolute, if $f^{-1}(\mathcal{B}) \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ for every $\mathcal{B} \in \mathfrak{N}\beta_{PC}o\left(\mathcal{H}, \tau_{RL^*{}_{eq}}(X)\right)$.

Definition 2.23.[10] Let $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ and $\left(\mathcal{H}, \tau^*_{RL_{eq}}(Y)\right)$ be two $\mathfrak{N}$.Top. S , a function $f: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{H}, \tau^*_{RL_{eq}}(Y)\right)$ is said to be Nano pre continuous, if the inverse image of every Nano open set in $\mathcal{H}$ is Nano pre-open in $\mathcal{M}$.

Definition 2.24.[8] Let $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ and $\left(\mathcal{H}, \tau^*_{RL_{eq}}(Y)\right)$ be two $\mathfrak{N}$.Top. S , a function $f: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{H}, \tau^*_{RL_{eq}}(Y)\right)$ is said to be Nano contra pre continuous , if the inverse image of every Nano open set in $\mathcal{H}$ is Nano pre-closed in $\mathcal{M}$

Definition 2.25.[11] Let $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ and $\left(\mathcal{H}, \tau^*_{RL_{eq}}(Y)\right)$ be two $\mathfrak{N}$.Top. S , a function $f: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{H}, \tau^*_{RL_{eq}}(Y)\right)$ is said to be Nano perfectly continuous if the inverse image of every Nano -open set in $\mathcal{H}$ is Nano clopen in $\mathcal{M}$.

Definition 2.26[11] Let $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ and $\left(\mathcal{H}, \tau^*_{RL_{eq}}(Y)\right)$ be two $\mathfrak{N}$.Top. S , a function $f: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{H}, \tau^*_{RL_{eq}}(Y)\right)$ is said to be Nano strongly continuous if the inverse image of every subset in $\mathcal{H}$ is Nano clopen in $\mathcal{M}$.

Definition 2.27.[13] A subset A of $\mathfrak{N}$.Top. S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ is said to be Nano δ-open if for each $x \in A$ there exist a Nano open set G such that $x \in G \subseteq \mathfrak{N}int\mathfrak{N}cl(G) \subseteq A$.

Proposition 2.28 [1] If $K \in \mathfrak{N}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ and $L \in \mathfrak{N}\beta o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$, then $K \cap L \in \mathfrak{N}\beta o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$

Proposition 2.29. If $K \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ and $L \in \mathfrak{N}CLP\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$, then $K \cap L \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$

Proof: Let $r \in K \cap L$, then $r \in K$ and $r \in L$ Since $K \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$, then there is $J \in \mathfrak{N}\mathrm{P}c\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ such that $r \in J \subseteq K$. Since $L \in \mathfrak{N}\mathrm{o}\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$, then by Proposition 2.28, $K \cap L \in \mathfrak{N}\mathrm{\beta o}\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$.

Also, $L \in \mathfrak{N}\mathrm{Pc}\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$, then $J \cap L \in \mathfrak{N}\mathrm{Pc}\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ such that $r \in J \cap L \subseteq K \cap L$. Hence $K \cap L \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$.

Definition 2.30.[8] Let $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ be a $\mathfrak{N}$.Top. S and let $K \subseteq M$, the set $\mathfrak{N}ker(K) = \cap\left\{H : K \subseteq H, H \in \mathfrak{N}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)\right\}$ is known as Nano kernel of K simply by $\mathfrak{N}ker(K)$.

Proposition 2.31.[8] Let $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ be a $\mathfrak{N}$.Top. S and let $K, L \subseteq \mathcal{M}$, then

J1) $p \in \mathfrak{N}ker(K)$ if and only if $K \cap O \neq \phi$ for any $O \in \mathfrak{N}c\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ containing p.

J2) If $K \subseteq L$, then $\mathfrak{N}ker(K) \subseteq \mathfrak{N}ker(L)$.

J3) If $K \in \mathfrak{N}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$, then $K = \mathfrak{N}ker(K)$.

3 Nano contra β_{PC}-continuous function

Definition 3.1 Let $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ and $\left(\mathcal{N}, \tau^*_{RL_{eq}}(Y)\right)$ be two $\mathfrak{N}$.Top. S , then a function $f : \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{N}, \tau^*_{RL_{eq}}(Y)\right)$ is named Nano contra β_{PC}-continuous if the inverse image of every Nano open set in $\mathcal{N}$ is Nano β_{PC}-closed set in $\mathcal{M}$.

Example 3.2. Let $H = \{a, b, c, d\}$ and $H/_R = \{\{a\}, \{b\}, \{c\}, \{d\}\}$, $X = \{c, d\}$, then $\tau_R(X) = \{H, \phi, \{c, d\}\}$. Assume that $\mathcal{M} = \{a, b, c, d\}$, equipped with $\mathcal{M}/_R = \{\{a, c\}, \{b\}, \{d\}\}$, $Y = \{b, d\}$, then $\tau_{R^*}(Y) = \{M, \phi, \{b, d\}\}$. Define function $g: \left(H, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{M}, \tau^*_{RL_{eq}}(Y)\right)$ as $g(a) = a$, $g(b) = b$, $g(c) = c$, $g(d) = d$, then g is Nano contra β_{PC}-continuous .

Lemma 3.3. Every Nano clopen is Nano β_{PC}-closed .

Proof : Let K be any Nano clopen subset of $\mathfrak{N}$.Top. S $\left(\mathcal{M}, \tau^*_{RL_{eq}}(Y)\right)$, then $M - K$ is also clopen subset of $\mathfrak{N}$.Top. S $\left(\mathcal{M}, \tau^*_{RL_{eq}}(Y)\right)$, it follows $M - K$ is β-open and pre-closed , then for any $p \in M - K$ we have $p \in M - K \subseteq M - K$ that is $M - K$ is Nano β_{PC}-open . Consequently K is Nano β_{PC}-closed .

Proposition 3.4. Every Nano strongly continuous is Nano contra β_{PC}-continuous and Nano β_{PC}-continuous.

Proof : Let $h : \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$ be Nano strongly continuous function and let $B \in \mathfrak{N}o\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$, then $h^{-1}(B)$ is a Nano clopen set in $\mathfrak{N}$.Top. S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$. By Lemma 3.3 , $h^{-1}(B) \in \mathfrak{N}\beta_{PC}c\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$. Hence h is Nano contra β_{PC}-continuous.

Remark 3.5. Nano contra β_{PC}-continuous and Nano β_{PC}-continuous functions are independent as showing in the following example

Example 3.6.

1-Consider Example 3.2 , g is Nano contra β_{PC}-continuous but not Nano β_{PC}-continuous

2. Let $\mathcal{M} = \{a, b, c, d, e\}$, $\mathcal{M}/_R = \{\{a, c\}, \{b\}, \{d\}, \{e\}\}$, $X = \{a, d, e\}$ then $\tau_R(X) = \{M, \phi, \{e, d\}, \{a, c\}, \{a, c, d, e\}\}$ and $\mathfrak{N} = \{x, y, z, w\}$, $\mathfrak{N}/_{R^*} = \{\{x\}, \{y, z\}, \{w\}\}$, $Y = \{x, z\}$, then $\tau_{R^*}(Y) = \{\mathfrak{N}, \phi, \{x\}, \{y, z\}, \{x, y, z\}\}$. Define $f: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{N}, \tau^*_{RL_{eq}}(Y)\right)$ as $f(a) = x$, $f(b) = w$, $f(c) = y$, $f(d) = z$, $f(e) = y$, then f is Nano β_{PC}-continuous but not Nano contra β_{PC}-continuous since $f^{-1}(\{x, y, z\}) = \{a, c, d, e\} \notin \mathfrak{N}\beta_{PC}c\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$.

Proposition 3.7. If φ is a mapping from $\mathfrak{N}$.Top. S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ into any Nano regular $\left(\mathfrak{N}, \tau_{RL_{eq}}(Y)\right)$ is Nano contra β_{PC}-continuous then φ is $\mathfrak{N}\beta_{PC}$-continuous function.

Proof: Let p be an arbitrary point of $\mathcal{M}$ and let H be a Nano open set containing $\varphi(p)$ in $\mathfrak{N}$. Since $\mathfrak{N}$ is Nano regular space, then there is $T \in \mathfrak{N}o\left(\mathfrak{N}, \tau_{RL_{eq}}(Y)\right)$ containing $\varphi(p)$ such that $\mathfrak{N}cl(T) \subseteq H$. Since φ is Nano contra β_{PC}-continuous, then by Proposition 2.7 there exists $K \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ containing p such that $\varphi(K) \subseteq \mathfrak{N}cl(T) \subseteq H$. Hence φ is $\mathfrak{N}\beta_{PC}$-continuous function.

Proposition 3.8 If φ is a mapping from Nano Pre Frechet space $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ into any $\mathfrak{N}$.Top. S $\left(\mathcal{N}, \tau_{RL_{eq}}(X)\right)$ is Nano contra β-continuous, then φ is Nano contra β_{PC}-continuous.

Proof : By proposition 2.9

Proposition 3.9 Every Nano contra β_{PC}-continuous is Nano contra β-continuous.

Proof : follows from Definition 2.10

Proposition 3.10 Every Nano pre-continuous and Nano contra β-continuous is Nano contra β_{PC}-continuous.

Proof : Let $\varphi: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$ be both Nano pre-continuous and Nano contra β-continuous ,, let $W \in \mathfrak{N}c\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$, since φ is Nano contra β-continuous , then $\varphi^{-1}(W) \in \mathfrak{N}\beta o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ again $W \in \mathfrak{N}c\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$, since φ is Nano pre-continuous , then $\varphi^{-1}(W) \in \mathfrak{N}Pc\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$. Consequently , $\varphi^{-1}(W) \in \mathfrak{N}\,\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$. Hence φ is Nano contra β_{PC}-continuous.

Definition 3.11 A $\mathfrak{N}$.Top. S $\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$ is said to be Nano locally β_{PC}-indiscrete if every $\mathfrak{N}\beta_{PC}$-open is Nano-closed

Proposition 3.12 If ψ is a mapping from Nano locally β_{PC}-indiscrete $\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$ into any $\mathfrak{N}$.Top. S $\left(\mathfrak{N},\tau_{RL_{eq}}(Y)\right)$ is Nano contra β_{PC}-continuous then ψ is $\mathfrak{N}$-continuous function.

Definition 3.13 A $\mathfrak{N}$.Top. S $\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$ is said to be Nano β_{PC}-space if every $\mathfrak{N}\beta_{PC}$-open is Nano-open.

Theorem 3.14. Let $\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$ and $\left(\mathfrak{N},\tau_{RL^{*}eq}(Y)\right)$ be two $\mathfrak{N}$.Top. S. For a function $\varpi:\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)\longrightarrow\left(\mathfrak{N},\tau_{RL^{*}eq}(Y)\right)$, then the following statements are equivalent .

E1) ϖ is Nano contra β_{PC}-continuous function.

E2) The inverse image of every Nano closed set in $\mathfrak{N}$ is Nanoβ_{PC}-open in $\mathcal{M}$.

E3) $\varpi(\mathfrak{N}\beta_{PC}cl(T)\subseteq\mathfrak{N}\ker(\varpi(T))$, for any subset T of $\mathcal{M}$.

E4) $\mathfrak{N}\beta_{PC}cl(\varpi^{-1}(L))\subseteq\varpi^{-1}(\mathfrak{N}\ker(L))$, for any subset L of $\mathfrak{N}$.

Proof : E1⇒E2 Let ϖ be Nano contra β_{PC}-continuous and let K is Nano closed set in $\mathfrak{N}$, then $\mathfrak{N}-K$ is Nano open. By utilizing E1) $\varpi^{-1}(\mathfrak{N}-K)$ is $\mathfrak{N}\beta_{PC}$-closed in $\mathcal{M}$, and since $\varpi^{-1}(\mathfrak{N}-K)=\mathcal{M}-\varpi^{-1}(K)$, then $\varpi^{-1}(K)$ is Nano β_{PC}-open in $\mathcal{M}$.

E2⇒E3 Assume that T be any subset of $\mathcal{M}$, suppose that $p\notin\mathfrak{N}\ker(\varpi(T))$ .By Proposition 2.31, there exists $H\in\mathfrak{N}c\left(\mathfrak{N},\tau_{RL^{*}eq}(Y)\right)$ having p such that $\varpi(T)\cap H=\phi$, so $T\cap\varpi^{-1}(H)=\phi$, that is $T\subseteq\mathcal{M}-\varpi^{-1}(H)$.

By applying E2, $\varpi^{-1}(H)\in\mathfrak{N}\beta_{PC}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$. It follows $\mathfrak{N}\beta_{PC}cl(T)\subseteq\mathcal{M}-\varpi^{-1}(H)\Rightarrow\mathfrak{N}\beta_{PC}cl(T)\cap\varpi^{-1}(H)=\phi\Rightarrow\varpi(\mathfrak{N}\beta_{PC}cl(T))\cap H=\phi$. Accordingly, $p\notin\varpi(\mathfrak{N}\beta_{PC}cl(T))$. Hence $\varpi(\mathfrak{N}\beta_{PC}cl(T)\subseteq\mathfrak{N}\ker(\varpi(T))$.

E3⇒E4 Let L be a subset of $\mathfrak{N}$.Top. S $\left(\mathfrak{N},\tau_{RL^{\star}eq}(Y)\right)$,then by applying E3 we have $\varpi(\mathfrak{N}\beta_{PC}cl(\varpi^{-1}(L))\subseteq\mathfrak{N}\ker\left(\varpi(\varpi^{-1}(L))\right)\subseteq\mathfrak{N}\ker(L)\Longrightarrow\mathfrak{N}\beta_{PC}cl(\varpi^{-1}(L))\subseteq\varpi^{-1}(\mathfrak{N}\ker(L))$.

E4⇒E1 Let $K\in\mathfrak{N}o\left(\mathfrak{N},\tau_{RL^{\star}eq}(Y)\right)$, then by applying E4, $\mathfrak{N}\beta_{PC}cl(\varpi^{-1}(K))\subseteq\varpi^{-1}(\mathfrak{N}\ker(K))$ and by Proposition 2.31 (3), $\varpi^{-1}(\mathfrak{N}\ker(K))=\varpi^{-1}(K)$ that is $\mathfrak{N}\beta_{PC}cl(\varpi^{-1}(K))\subseteq\varpi^{-1}(K)$. Hence ϖ is Nano contra β_{PC}-continuous function .

Proposition 3.15 If ψ is a mapping from Nano β_{PC}-space $\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$ into any $\mathfrak{N}$.Top. S $\left(\mathfrak{N},\tau_{RL_{eq}}(Y)\right)$ is Nano contra β_{PC}-continuous then ψ is $\mathfrak{N}$ contra continuous function.

Proposition 3.16 If $\varphi:\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)\longrightarrow\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)$ is Nano β_{PC}-Irresolute and $\psi:\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)\rightarrow\left(\mathrm{T},\tau^{\#}_{RL_{eq}}(Z)\right)$ is Nano contra β_{PC}-continuous, then $\psi\circ\varphi:\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)\longrightarrow\left(\mathrm{T},\tau^{\#}_{RL_{eq}}(Z)\right)$ is Nano contra β_{PC}-continuous.

Proof: Let $K \in \mathfrak{N}O\left(T, \tau^{\#}_{RL_{eq}}(Z)\right)$. Since ψ is Nano contra β_{PC}-continuous, then $\psi^{-1}(K) \in \mathfrak{N}\beta_{PC}C\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ and since φ is Nano β_{PC}-Irresolute function, then $\varphi^{-1}\left(\psi^{-1}(K)\right) = (\psi \circ \varphi)^{-1}(K) \in \mathfrak{N}\beta_{PC}C\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$. Hence $\psi \circ \varphi$ is Nano contra β_{PC}-continuous.

Proposition 3.17 If $\varphi: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^{*}_{RL_{eq}}(Y)\right)$ is Nano contra β_{PC}-continuous and $\psi: \left(\mathfrak{N}, \tau^{*}_{RL_{eq}}(Y)\right) \to \left(T, \tau^{\#}_{RL_{eq}}(Z)\right)$ is Nano continuous,

then $\psi \circ \varphi: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(T, \tau^{\#}_{RL_{eq}}(Z)\right)$ is Nano contra β_{PC}-continuous.

4 Nano almost contra β_{PC}-continuous function

Definition 4.1. A function $\varphi: \left(\mathcal{U}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{V}, \tau_{RL^{\star}{}_{eq}}(Y)\right)$ is named Nano almost contra β_{PC} −continuous if $\varphi^{-1}(K) \in \mathfrak{N}\beta_{PC}c\left(\mathcal{U}, \tau_{RL_{eq}}(X)\right)$ for any $K \in \mathfrak{N}\mathcal{R}o\left(\mathcal{V}, \tau_{RL^{\star}{}_{eq}}(Y)\right)$.

Proposition 4.2. Every Nano contra β_{PC} −continuous function is Nano almost contra β_{PC} − continuous .

Proof : Let $\varphi: \left(\mathcal{U}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{V}, \tau_{RL^{\star}{}_{eq}}(Y)\right)$ be Nano contra β_{PC} − continuous and let $K \in \mathfrak{N}\mathcal{R}o\left(\mathcal{V}, \tau_{RL^{\star}{}_{eq}}(Y)\right)$, then $K \in \mathfrak{N}o\left(\mathcal{V}, \tau_{RL^{\star}{}_{eq}}(Y)\right)$. But φ is Nano contra β_{PC} -continuous, thus $\varphi^{-1}(K) \in \mathfrak{N}\beta_{PC}c\left(\mathcal{U}, \tau_{RL_{eq}}(X)\right)$. Hence φ is Nano almost contra β_{PC} -continuous .

The converse of Proposition 4.2 may not be true as showing in the following example.

Example 4.3 consider the same set an Example 3.6.(2). Let $\mathcal{M} = \{a, b, c, d, e\}$, $\mathcal{M}/_{R} = \{\{a, c\}, \{b\}, \{d\}, \{e\}\}$, $X = \{a, d, e\}$ then $\tau_R(X) = \{M, \phi, \{e, d\}, \{a, c\}, \{a, c, d, e\}\}$ and $\mathfrak{N} = \{x, y, z, w\}$, $\mathfrak{N}/_{R^*} = \{\{x\}, \{y, z\}, \{w\}\}$, $Y = \{x, z\}$, then $\tau_{R^*}(Y) = \{\mathfrak{N}, \phi, \{x\}, \{y, z\}, \{x, y, z\}\}$. Define $f: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{N}, \tau^{*}_{RL_{eq}}(Y)\right)$ as $f(a) = x$, $f(b) = w$, $f(c) = y$, $f(d) = z$, $f(e) = y$, then f is Nano almost contra β_{PC} -continuous but not Nano contra β_{PC} -continuous

Proposition 4.4. Let a function $\varphi: \left(\mathcal{U}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{V}, \tau_{RL^{\star}{}_{eq}}(Y)\right)$ be Nano almost contra β_{PC} -continuous if and only if for each $p \in \mathcal{U}$ and any K is $\mathfrak{N}Rc$-set of $\mathcal{V}$ containing $\varphi(p)$, there exists a Nano β_{PC}-open set T containing p such that $\varphi(T) \subseteq K$.

Proof : Suppose that φ is Nano almost contra P_{PC}-continuous . Let $K \in \mathfrak{N}\mathcal{R}c\left(\mathcal{V}, \tau_{RL^{\star}{}_{eq}}(Y)\right)$ containing $\varphi(p)$ for some $p \in \left(\mathcal{U}, \tau_{RL_{eq}}(X)\right)$, then $\mathcal{V} - K \in \mathfrak{N}\mathcal{R}o\left(\mathcal{V}, \tau_{RL^{\star}{}_{eq}}(Y)\right)$. since φ is Nano almost contra P_{PC}-continuous , then $\varphi^{-1}(\mathcal{V} - K) = \mathcal{U} - \varphi^{-1}(K) \in \mathfrak{N}P_{PC}c\left(\mathcal{U}, \tau_{RL_{eq}}(X)\right)$,consequently , $\varphi^{-1}(K) \in \mathfrak{N}P_{PC}o\left(\mathcal{U}, \tau_{RL_{eq}}(X)\right)$

containing p . set $\varphi^{-1}(K) = T$, then $\varphi(T) = \varphi(\varphi^{-1}(K) \subseteq K$ where $T \in \mathfrak{N}P_{PC}o\left(\mathcal{U}, \mathcal{T}_{RL_{eq}}(X)\right)$ containing p .

Proposition 4.4. For a function $\varphi: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$, then the following are equivalent .

E1) φ is Nano almost contra β_{PC}-continuous .

E2) $\varphi^{-1}(K) \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$, for any $K \in \mathfrak{N}\theta So\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$.

E3) $\varphi^{-1}(L) \in \mathfrak{N}\beta_{PC}\, int\left(\varphi^{-1}\left(\mathfrak{N}cl(L)\right)\right)$, for any $L \in \mathfrak{N}So\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$.

Proof : E1⟹E2 . Let $K \in \mathfrak{N}\theta So\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$, then for every $p \in K$, there exists $G \in \mathfrak{N}So\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$ such that $p \in G \subseteq \mathfrak{N}cl(G) \subseteq K$. Accordingly , $K = \bigcup \mathfrak{N}cl\mathfrak{N}int(G_i)$, it follows that $\varphi^{-1}(K) = \varphi^{-1}\left(\bigcup_i \mathfrak{N}cl\mathfrak{N}int(G_i)\right) = \bigcup_i \varphi^{-1}\left(\mathfrak{N}cl\mathfrak{N}int(G_i)\right)$. By E1 , φ is Nano almost contra β_{PC}-continuous , then by Proposition 2.11 , $\varphi^{-1}\left(\mathfrak{N}cl\mathfrak{N}int(G_i)\right) \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ for each i .That is $\varphi^{-1}(K) \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$.

E2⟹E3 . Let $L \in \mathfrak{N}So\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$, then $\mathfrak{N}cl(L) \in \mathfrak{N}Rc\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$. Accordingly , $\mathfrak{N}cl(L) \in \mathfrak{N}\theta So\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$. By applying E2 $\varphi^{-1}\left(\mathfrak{N}cl(L)\right) \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$. This implies that $\varphi^{-1}(L) \subseteq \varphi^{-1}\left(\mathfrak{N}cl(L)\right) = \mathfrak{N}\beta_{PC}\, int\left(\varphi^{-1}\left(\mathfrak{N}cl(L)\right)\right)$.

E3⟹E1 . Let $H \in \mathfrak{N}Rc\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$, then $H \in \mathfrak{N}\theta So\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$, by applying E3 $\varphi^{-1}(H) \subseteq \mathfrak{N}\beta_{PC}\, int\left(\varphi^{-1}\left(\mathfrak{N}cl(H)\right)\right) = \mathfrak{N}\beta_{PC}\, int\left(\varphi^{-1}(H)\right)$. Hence φ is Nano almost contra β_{PC}-continuous function .

Proposition 4.5. For a function $\varphi: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$, then the following are equivalent .

E1) φ is Nano almost contra β_{PC}-continuous function .

E2) $\mathfrak{N}\beta_{PC}\, cl\left(\varphi^{-1}(K)\right) \subseteq \varphi^{-1}\left(\mathfrak{N}Scl(K)\right)$ for any $K \in \mathfrak{N}o\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$.

E3) $\mathfrak{N}\beta_{PC}\, cl\left(\varphi^{-1}(H)\right) \subseteq \varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(H)\right)$ for $H \in \mathfrak{N}o\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$.

Proof : E1⟹E2 . Let $K \in \mathfrak{N}o\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$, since $\mathfrak{N}Scl(K) = K \bigcup \mathfrak{N}int\mathfrak{N}cl(K) = \mathfrak{N}int\mathfrak{N}cl(K)$, then $\mathfrak{N}Scl(K) \in \mathfrak{N}Ro\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$. By applying E1 , $\varphi^{-1}\left(\mathfrak{N}Scl(K)\right) \in \mathfrak{N}\beta_{PC}c\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ and since $\varphi^{-1}(K) \subseteq \varphi^{-1}\left(\mathfrak{N}Scl(K)\right)$, then $\mathfrak{N}\beta_{PC}\, cl\left(\varphi^{-1}(K)\right) \subseteq \varphi^{-1}\left(\mathfrak{N}Scl(K)\right)$.

E2⟹E3 . Follows directly from that fact $\mathfrak{N}Scl(A) = \mathfrak{N}int\mathfrak{N}cl(A)$ for any .

E3⟹E1 . Let $H \in \mathfrak{N}Ro\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$, then $H \in \mathfrak{N}o\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$. By applying E3 $\mathfrak{N}\beta_{PC}\, cl\left(\varphi^{-1}(H)\right) \subseteq \varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(H)\right) = \varphi^{-1}(H)$. Consequently , $\varphi^{-1}(H) \in \mathfrak{N}\beta_{PC}c\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$. Hence φ is Nano almost contra β_{PC}-continuous function .

5 Nano almost β_{PC} -continuous function .

Definition 5.1. A function $f:\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)\longrightarrow\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)$ is Nano almost β_{PC} -continuous if and only if the inverse image of any Nano regular open set in $\mathfrak{N}$ is Nano β_{PC}-open in $\mathcal{M}$.

Proposition 5.2. A function $f:\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)\longrightarrow\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)$ is Nano almost β_{PC} -continuous if and only if for each $r\in M$ and each $K\in\mathfrak{N}o\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)$ containing $f(r)$, there exists $L\in\mathfrak{N}\beta_{PC}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$ containing r such that $f(L)\subseteq\mathfrak{N}int\mathfrak{N}cl(K)$.

Proof : Let $K\in\mathfrak{N}o\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)$ containing $f(r)$, for $r\in M$, then $\mathfrak{N}int\mathfrak{N}cl(K)\in\mathfrak{N}Ro\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)$, since f is Nano almost β_{PC} –continuous . $f^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(K)\right)\in\mathfrak{N}\beta_{PC}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$ such that $r\in f^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(K)\right)=L$, then $L\in\mathfrak{N}\beta_{PC}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$ and $L\subseteq f^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(K)\right)$ consequently , $f(L))\subseteq\mathfrak{N}int\mathfrak{N}cl(K)$. Conversely , let $K\in\mathfrak{N}Ro\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)$, then $K\subseteq\mathfrak{N}int\mathfrak{N}cl(K)$, by hypothesis there exists $L\in\mathfrak{N}\beta_{PC}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$ such that $f(L))\subseteq\mathfrak{N}int\mathfrak{N}cl(K)$,then $L\subseteq f^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(K)\right)\Longrightarrow L=f^{-1}(K)$, $f^{-1}(K)\in\mathfrak{N}\beta_{PC}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$. Hence f is Nano almost β_{PC} -continuous function .

Proposition 5.3. A function $f:\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)\longrightarrow\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)$ is Nano almost β_{PC} -continuous if and only if the inverse image of any Nano regular closed set in $\mathfrak{N}$ is Nano β_{PC}-closed in $\mathcal{M}$.

Proposition 5.4. Every Nano β_{PC} -continuous is Nano almost β_{PC} -continuous .

Proof : Let $\varphi:\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)\longrightarrow\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)$ be Nano β_{PC} -continuous , let $A\in\mathfrak{N}Ro\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)$, then $A\in\mathfrak{N}\beta o\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)$ since φ is Nano β_{PC} -continuous . $\varphi^{-1}(A)\in\mathfrak{N}\beta_{PC}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$. Hence φ is Nano almost β_{PC} -continuous function .

Proposition 5.5. A function $\varphi:\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)\longrightarrow\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)$ is Nano almost β_{PC} -continuous if and only if the inverse image of every Nano δ-open set in $\mathfrak{N}$ is Nano β_{PC}-open in $\mathcal{M}$.

Proof : Suppose φ is Nano almost β_{PC} -continuous and let $H\in\mathfrak{N}\delta o\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)$. To show that $\varphi^{-1}(H)\in\mathfrak{N}\beta_{PC}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$, if $\varphi^{-1}(H)=\phi$, then the proof is done . if not , let $x\in\varphi^{-1}(H)$, then $\varphi(x)\in H$ and since $H\in\mathfrak{N}\delta o\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)$, then there is $L\in\mathfrak{N}o\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)$ such that $\varphi(x)\in L\subseteq\mathfrak{N}int\mathfrak{N}cl(L)\subseteq H$, since $\mathfrak{N}int\mathfrak{N}cl(L)\in\mathfrak{N}Ro\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)$ and since φ is Nano almost β_{PC} -continuous , then $\varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(L)\right)\in\mathfrak{N}\beta_{PC}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$, consequently $\varphi^{-1}(H)\in\mathfrak{N}\beta_{PC}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$.

Conversely , suppose that the inverse image of every Nano δ-open set in $\mathfrak{N}$ is Nano β_{PC}-open in $\mathcal{M}$, to show that φ is Nano almost β_{PC} -continuous . Let $K\in\mathfrak{N}o\left(\mathfrak{N},\tau^{*}_{RL_{eq}}(Y)\right)$

containing $\varphi(x)$, then $\mathfrak{N}int\mathfrak{N}cl(K) \in \mathfrak{N}\delta o\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$, by hypothesis $\varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(K)\right) \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ containing x . Accordingly , $\varphi\left(\varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(K)\right)\right) \subseteq \mathfrak{N}int\mathfrak{N}cl(K)$, set $\varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(K)\right) = U \Rightarrow \varphi(U) \subseteq \mathfrak{N}int\mathfrak{N}cl(K)$, thus φ is Nano almost β_{PC} -continuous function .

Proposition 5.6. A function $\varphi:\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$ is Nano almost β_{PC} -continuous if and only if $\varphi:\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^{*S}_{RL_{eq}}(Y)\right)$ Nano β_{PC} -continuous .

Proof : Suppose that $\varphi:\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$ is Nano almost β_{PC} -continuous , and let $K \in \left(\mathfrak{N}, \tau^{*S}_{RL_{eq}}(Y)\right)$, then $K \in \mathfrak{N}Ro\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$, then $\varphi^{-1}(K) \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$. That is $\varphi:\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^{*S}_{RL_{eq}}(Y)\right)$ Nano β_{PC} -continuous .

Conversely , let $\varphi:\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^{*S}_{RL_{eq}}(Y)\right)$ Nano β_{PC} -continuous. Let $H \in \mathfrak{N}o\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$, containing $\varphi(X)$, thus $\mathfrak{N}int\mathfrak{N}cl(H) \in \mathfrak{N}Ro\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$ containing $\varphi(X)$. Accordingly $\mathfrak{N}int\mathfrak{N}cl(H) \in \left(\mathfrak{N}, \tau^{*S}_{RL_{eq}}(Y)\right)$, since $\varphi:\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^{*S}_{RL_{eq}}(Y)\right)$ Nano β_{PC} -continuous , then $\varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(H)\right) \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \Longrightarrow \varphi\left(\varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(H)\right)\right) \subseteq \mathfrak{N}int\mathfrak{N}cl(H)$, let $\varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(H)\right) = U$. Hence $\varphi:\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$ is Nano almost β_{PC} -continuous .

Remark 5.7. A N .Top .S $\left(\mathfrak{N}, \tau^{*S}_{RL_{eq}}(Y)\right)$ is the family of all Nano regular open set .

Definition 5.8. A N .Top .S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ is said to be Nano Hyper connected if $\mathfrak{N}cl(G) = M$,for any $G \in \mathfrak{N}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$.

Proposition 5.9. If $\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$ is Nano Hyper connected , then every map $\varphi:\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$ is Nano almost β_{PC} -continuous .

Proof : Let $\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$ is Nano Hyper connected , let $K \in \mathfrak{N}o\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$ containing $\varphi(X)$, then $\mathfrak{N}cl(K) = \mathfrak{N}$. Consequently $\mathfrak{N}int\mathfrak{N}cl(K) = \mathfrak{N}$. That is for any $L \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ containing x , $\varphi(L) \subseteq \mathfrak{N} = \mathfrak{N}int\mathfrak{N}cl(K)$. Hence φ is Nano almost β_{PC} -continuous .

Theorem 5.10. A function $\varphi:\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$ is Nano almost β_{PC} -continuous if and only if for any $K \in \mathfrak{N}o\left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$, $\varphi^{-1}(K) \subseteq \mathfrak{N}\beta_{PC} - int\left(\varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(K)\right)\right)$.

Proof : Suppose that φ is Nano almost β_{PC} -continuous and let $K \in \mathfrak{N}o\left(\mathfrak{N}, \tau^{*}_{RL_{eq}}(Y)\right)$. Let $x \in \varphi^{-1}(K)$, then $\varphi(X) \in K \in \mathfrak{N}o\left(\mathfrak{N}, \tau^{*}_{RL_{eq}}(Y)\right)$, since φ is Nano almost β_{PC} -continuous , then there is $L \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ containing x such that $\varphi(L) \subseteq \mathfrak{N}int\mathfrak{N}cl(K)$. Consequently , $x \in L \subseteq \varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(K)\right)$, it follows $\varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(K)\right) \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$. That is $x \in \mathfrak{N}\beta_{PC} - int\left(\varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(K)\right)\right)$. Hence $\varphi^{-1}(K) \subseteq \mathfrak{N}\beta_{PC} - int\left(\varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(K)\right)\right)$.

Conversely , suppose that $\varphi^{-1}(K) \subseteq \mathfrak{N}\beta_{PC} - int\left(\varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(K)\right)\right)$ for any $K \in \mathfrak{N}o\left(\mathfrak{N}, \tau^{*}_{RL_{eq}}(Y)\right)$. To show that φ is Nano almost β_{PC} -continuous , let $x \in \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ and let $L \in \mathfrak{N}o\left(\mathfrak{N}, \tau^{*}_{RL_{eq}}(Y)\right)$ containing $\varphi(x)$, then by hypothesis $x \in \varphi^{-1}(L) \subseteq \mathfrak{N}\beta_{PC} - int\left(\varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(L)\right)\right) \subseteq \varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(L)\right)$, take $\mathfrak{N}\beta_{PC} - int\left(\varphi^{-1}\left(\mathfrak{N}int\mathfrak{N}cl(L)\right)\right) = U$. That is $\varphi(U) \subseteq \mathfrak{N}int\mathfrak{N}cl(L)$. Hence φ is Nano almost β_{PC} -continuous .

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