A note on conjugate Bayesian estimators of random effects model

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Abstract. The theoretical findings for the Bayes random effects model and the Bayes random effects model with linear constraints on the model coefficients are the contribution of this study. We take into account the random effect model, which includes both fixed and random effects in addition to the experimental error term. We sought to offer a detailed examination of some characteristics of the Bayes and restricted Bayes estimators of the model in addition to applying the Bayesian approach to draw conclusions about the model using a conjugate prior distribution.

1 Introduction

The tools of probability theory are used in Bayesian analysis to convey uncertainty over the parameter's worth. The credibility of probable values for parameters is currently reflected by their density functions. The Bayesian technique aims to update parameter uncertainty distributions using data, and then utilize these updated distributions to make rational decisions. Closed-form density functions of the conditional posterior distributions of the parameters are still available even though closed-form marginal posterior distributions can be calculated when the prior distributions are conjugate. With the current draws for the other parameters and the conditional distributions of each parameter, Gibbs sampling can be employed in the case of conjugate priors [1].

Prior knowledge of all or some important parameters is one feature that distinguishes restricted models from other types. As an illustration, we could know the sign of the coefficients, that the sum of the coefficients is zero, or even that some coefficients are arranged in a certain order. Due to the parameters' narrower range as a result of this information, effective estimation techniques are needed. This paper provided references on Bayesian estimation method. Berger [2], Kori and AL-Mouel[3], Al-Isawi, AL-Mouel and Ali[4], Oloyede[5],[6], Lewis, MacEachern and Lee[7], Bhattacharya, Pati and Yang [8], Jewson, Smith and Holmes [9], Sosa and Aristizabal[10], Alquier[11], Mohaisen and Swadi[12], Yuan, Xiang and Zhang[13], Castillo-Mateo, Asín, Cebrián, Mateo-Lázaro and Abaurrea [14], .

The theoretical findings for the Bayes random effects model and the Bayes random effects model with linear constraints on the model coefficients are the contribution of this study. We take into account the random effect model, which includes both fixed and random effects in addition to the experimental error term. Using a conjugate prior, the Bayesian technique is used to draw conclusions about the model. We also tried to offer a detailed study of some characteristics of the Bayes and restricted Bayes estimators of the model.

The five components that make up this essay's structure and substance are as follows. We cover the random effects model, which includes the experimental error term, fixed and random effects, in the section that follows this. For the random effects model, we also offer theoretical results. The third section examined various effects of the conjugate Bayesian estimator for the study's model, which enables us to include linear constraints with the conjugate Bayesian estimator. Fourth, we examine the constrained Bayesian estimator's consistency property. For the purpose of demonstrating the efficacy of our methodology, part five presents a simulation study. Following the discussion of the conclusions.

2 The model and prior distribution

This paper proposes the following model:

$$Y_{NT\times 1} = F_{NT\times(K+1)}\theta_{(K+1)\times 1} + \omega_{NT\times 1}, \quad (1)$$

$$\omega = u + \varepsilon \Longrightarrow \omega \sim N(0, \Psi), \Longrightarrow Y \sim N_{NT}(F\theta, \Psi),$$

$$\Psi = \sigma_\varepsilon^2(\mathrm{I}_N \otimes E_T) + (\sigma_\varepsilon^2 + T\sigma_u^2)(\mathrm{I}_N \otimes \mathrm{J}_T) = \sigma_\varepsilon^2 Q + \sigma_1^2 P,$$

where, $\sigma_1^2 = (\sigma_\varepsilon^2 + \mathrm{T}\sigma_u^2)$, $\mathrm{J}_\mathrm{T} = \frac{1}{T}\mathbf{1}\mathbf{1}^T$ and $\mathrm{E}_\mathrm{T} = \mathrm{I}_\mathrm{T} - \mathrm{J}_\mathrm{T}$.

The joint density of the $Y's$ serves as the likelihood function, wherein

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$$L(Y;\theta,\Psi) = (2\pi)^{\frac{-NT}{2}}|\Psi|^{\frac{-1}{2}} \exp\{\tfrac{-1}{2}(Y-F\theta)^T\Psi^{-1}(Y-F\theta)\} = (2\pi)^{\frac{-NT}{2}}(\sigma_\varepsilon^2)^{\frac{-N(T-1)}{2}}(\sigma_1^2)^{\frac{-N}{2}} \exp\{\tfrac{-1}{2}\left(Y-F\theta\right)^T\left[\tfrac{Q}{\sigma_\varepsilon^2}+\tfrac{P}{\sigma_1^2}\right](Y-F\theta)\}.$$

Following that, the parameter likelihood estimators are,[15]

$$\left.\begin{aligned} \hat{\theta} &= (F^T\Psi^{-1}F)^{-1}(F^T\Psi^{-1}Y), \\ \hat{\sigma}_\varepsilon^2 &= \tfrac{1}{N(T-1)}\left(Y-F\hat{\theta}\right)^T Q\left(Y-F\hat{\theta}\right), \\ and\quad \hat{\sigma}_1^2 &= \tfrac{1}{N}\left(Y-F\hat{\theta}\right)^T P\left(\left(Y-F\hat{\theta}\right)\right). \end{aligned}\right\}. \qquad (2)$$

We require a prior distribution on $(\theta,\sigma_\varepsilon^2,\sigma_1^2)$ to construct a full Bayesian model. We will use $N_{K+1}(\mu_\theta, \Sigma)$, of the vector parameters θ, and inverse gamma with parameters α_ε, β_ε, α_1 and β_1 is taken to be the prior distribution on σ_ε^2 and σ_1^2 correspondingly. i.e.

$\pi_0(\theta) = (2\pi)^{\frac{-k^*}{2}}\Sigma^{\frac{-1}{2}}\exp\{\frac{-1}{2}(\theta-\mu_\theta)^T\Sigma^{-1}(\theta-\mu_\theta)\}$, where $K^* = K+1$,

$\pi_0(\sigma_\varepsilon^2) = \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)}(\sigma_\varepsilon^2)^{-(\alpha_\varepsilon+1)}\exp\left(-\frac{\beta_\varepsilon}{\sigma_\varepsilon^2}\right)$, and $\pi_0(\sigma_1^2) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}(\sigma_1^2)^{-(\alpha_1+1)}\exp\left(-\frac{\beta_1}{\sigma_1^2}\right)$,

where α_ε, β_ε, The statistician must select the hyperparameters α_1 and β_1, which determine the priors.

3 Conjugate Bayesian estimator of the model

This section examines the implications of the conjugate Bayesian estimator on the model (1), enabling us to combine the conjugate Bayesian estimator with linear constraints. The posterior distribution of θ is:

$$\pi_1(\theta\,|\,Y) \propto L(Y|\sigma_\varepsilon^2,\sigma_1^2)\times\pi_0(\theta,\sigma_\varepsilon^2,\sigma_1^2)$$

$$\propto (2\pi)^{\frac{-NT}{2}}(\sigma_\varepsilon^2)^{\frac{-N(T-1)}{2}}(\sigma_1^2)^{\frac{-N}{2}}\exp\left\{-\tfrac{1}{2}(Y-F\theta)^T\Psi^{-1}(Y-F\theta)\right\}$$

$$|\Sigma|^{-\frac{1}{2}}\exp\{-\frac{1}{2}(\theta-\mu_\theta)^T\Sigma^{-1}(\theta-\mu_\theta)\}$$

$$\frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)}(\sigma_\varepsilon^2)^{-(\alpha_\varepsilon+1)}\exp\{\frac{-\beta_\varepsilon}{\sigma_\varepsilon^2}\}\frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}(\sigma_1^2)^{-(\alpha_1+1)}\exp\{\frac{-\beta_1}{\sigma_1^2}\}$$

$$\propto (2\pi)^{\frac{-NT}{2}}(\sigma_1^2)^{-\left(\alpha_1+\frac{N}{2}+1\right)}(\sigma_\varepsilon^2)^{-\left(\alpha_\varepsilon+\frac{N(T-1)}{2}+1\right)}\exp\{-\frac{1}{2}\left(Y-F\hat{\theta}\right)^T$$

$$\Psi^{-1}\left(Y-F\hat{\theta}\right)+(\theta-\mu_\theta)^T\Sigma^{-1}(\theta-\mu_\theta)+\beta^*\}, \qquad (3)$$

where $\beta^* = \dfrac{\beta_\varepsilon}{\sigma_\varepsilon^2}+\dfrac{\beta_1}{\sigma_1^2}$.

The following can be deduced from the exponent of (3):

$$\begin{aligned}(Y-F\theta)^T\Psi^{-1}(Y-F\theta)&+(\theta-\mu_\theta)^T\Sigma^{-1}(\theta-\mu_\theta)+\beta^* \\ &= Y^T\Psi^{-1}Y^T-2\theta^TF^T\Psi^{-1}Y+\theta^TF^T\Psi^{-1}F\theta+\theta^T\Sigma^{-1}\theta-2\theta^T\Sigma^{-1}\mu_\theta+\mu_\theta^T\Sigma^{-1}\mu_\theta+\beta^* \\ &= \theta^T(F^T\Psi^{-1}F+\Sigma^{-1})\,\theta-2\,\theta^T(F^T\Psi^{-1}Y+\Sigma^{-1}\mu_\theta)+Y^T\Psi^{-1}Y^T+\mu_\theta^T\Sigma^{-1}\mu_\theta \\ &\quad+\mu_\theta^{*T}(F^T\Psi^{-1}F+\Sigma^{-1})\mu_\theta^*-\mu_\theta^{*T}(F^T\Psi^{-1}F+\Sigma^{-1})\mu_\theta^*+\beta^* \\ &= (\theta-\mu_\theta^*)^T(F^T\Psi^{-1}F+\Sigma^{-1})(\theta-\mu_\theta^*)+sum+\beta^*,\end{aligned}$$

where,

$sum = Y^T\Psi^{-1}Y^T+\mu_\theta^T\Sigma^{-1}\mu_\theta-\mu_\theta^{*T}(F^T\Psi^{-1}F+\Sigma^{-1})\mu_\theta^*$,

$\mu_\theta^* = (F^T\Psi^{-1}F+\Sigma^{-1})^{-1}(F^T\Psi^{-1}Y+\Sigma^{-1}\mu_\theta)$.

Then

$$\pi_1(\theta\,|\,Y) \propto exp\,\{-\tfrac{1}{2}(\theta-\mu_\theta^*)^T(F^T\Psi^{-1}F+\Sigma^{-1})(\theta-\mu_\theta^*)\}, \qquad (4)$$

that is

$\theta \sim N_{K^*}(\mu_\theta^*, (F^T\Psi^{-1}F+\Sigma^{-1})^{-1})$. Thus the Bayes estimator of θ is

$$\breve{\theta} = \mu_\theta^* = (F^T\Psi^{-1}F+\Sigma^{-1})^{-1}(F^T\Psi^{-1}Y+\Sigma^{-1}\mu_\theta). \qquad (5)$$

Now, to obtain the Bayesian estimator of θ with linear constraints, $R\theta = r$, where R is $m\times k^*$ and r is $m\times 1$ we use the form of Lagrangian function

$$\Omega = (\theta^c-\breve{\theta})^T(F^T\Psi^{-1}F+\Sigma^{-1})\left(\theta^c-\breve{\theta}\right)+\lambda^T(R\theta-r), \qquad (6)$$

where λ is $(m\times 1)$ vector of Lagrangian multipliers. Now, differentiation (6) above with respect to θ^c and λ we can get

$$2(F^T\Psi^{-1}F+\Sigma^{-1})\theta^c-2(F^T\Psi^{-1}F+\Sigma^{-1})\breve{\theta}+R^T\lambda = 0, \qquad (7)$$

$$R\theta-r = 0\,. \qquad (8)$$

Multiply equation (7) by $R(F^T\Psi^{-1}F+\Sigma^{-1})^{-1}$ to obtain

$$2R\theta^c - 2R\breve{\theta} + \lambda\, R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T = 0 \quad (9)$$

$$\rightarrow \lambda[R(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T] = 2R\breve{\theta} - 2R\theta^c$$

$$\rightarrow \lambda = [R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1} \times 2(R\breve{\theta} - r)$$

$$\therefore \lambda = -2\,[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}(r - R\breve{\theta})\,. \quad (10)$$

Now, substitute (10) in equation (7) to get

$$2(F^T\Psi^{-1}F + \Sigma^{-1})\,\theta^c - 2(F^T\Psi^{-1}F + \Sigma^{-1})\breve{\theta} - 2R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}\big(r - R\breve{\theta}\big) = 0$$

$$\rightarrow 2(F^T\Psi^{-1}F + \Sigma^{-1})\,\theta^c = 2(F^T\Psi^{-1}F + \Sigma^{-1})\breve{\theta} + 2R^T[R\,(F^T\Psi^{-1}F+\Sigma^{-1})^{-1}R^T]^{-1}(\,r - R\breve{\theta})$$

$$\rightarrow\ \theta^c = \breve{\theta} + (F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}\big(r - R\breve{\theta}\big). \quad (11)$$

4 Asymptotic properties

We look into the constrained Bayesian estimator's consistency property in this section.

Theorem 1: If Y is normal distribution with $E(Y) = F\theta$, and $cov(Y) = \Psi$, $R\theta = r$ and $\lim_{NT\to\infty}\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}$ is finite positive definite. Then, $p\lim_{NT\to\infty}(\,\theta^c) = \theta$.

Proof:

$$p\lim_{NT\to\infty}(\,\theta^c\,) = p\lim_{NT\to\infty}\{\breve{\theta} + (F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}\big(r - R\breve{\theta}\big)\}$$

$$\Rightarrow p\lim_{NT\to\infty}(\,\theta^c) = p\lim_{NT\to\infty}(\breve{\theta}) + \lim_{NT\to\infty}\left(\frac{1}{NT}F^T\Psi^{-1}F + \frac{1}{NT}\Sigma^{-1}\right)^{-1}R^T\left[R\lim_{NT\to\infty}\left(\frac{1}{NT}F^T\Psi^{-1}F + \frac{1}{NT}\Sigma^{-1}\right)^{-1}R^T\right]^{-1}(r - p\lim_{NT\to\infty}R\breve{\theta})\,,$$

$$\Rightarrow p\lim_{NT\to\infty}(\,\theta^c) = p\lim_{NT\to\infty}(\breve{\theta}) + \lim_{NT\to\infty}\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T \left[R\lim_{NT\to\infty}\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T\right]^{-1}(r - p\lim_{NT\to\infty}R\breve{\theta})\,,$$

$$= \theta + \lim_{NT\to\infty}\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T + \left[R\lim_{NT\to\infty}\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T\right]^{-1}(r - R\theta)\,.$$

$\because R\theta = r,$

$\therefore p\lim_{NT\to\infty}(\,\theta^c) = \theta.$ ■

Theorem 2: Let $Y \sim N_{NT}(F\theta, \Psi)$, $R\theta = r$, and F is a nonsingular matrix, then the restricted Bayesian estimator θ^c is asymptotically normally distributed with $E(\theta^c) = 0$ and $cov(\theta^c) = \xi$,

$$\xi = \{\left(\tfrac{1}{NT}F^T\Psi^{-1}F\right)^{-1} - \left(\tfrac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T[R\,\left(\tfrac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T]R\left(\tfrac{1}{NT}F^T\Psi^{-1}F\right)^{-1}\}$$

Proof:

Since $\breve{\theta} = (F^T\Psi^{-1}F + \Sigma^{-1})^{-1}(F^T\Psi^{-1}Y + \Sigma^{-1}\mu_\theta)$, if $NT \to \infty$,then $\breve{\theta} \to \hat{\theta}$ thus , by central limit theorem we have $\sqrt{NT}\big(\breve{\theta} - \theta\big) \xrightarrow{d} N[0, \frac{1}{NT}(F^T\Psi^{-1}F)^{-1}]$.

$$\because \theta^c = \breve{\theta} + (F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}(r - R\breve{\theta})$$

$$= \breve{\theta} - (F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}\,R\breve{\theta} + (F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}\,r$$

$$= [\mathrm{I} - (F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}\,R\,]\,\breve{\theta} + [(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}R]\theta.$$

From theorem 1,we get

$$\theta^c = [\mathrm{I} - (F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}\,R\,]\,\breve{\theta} + [\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}R]\,\theta$$

$$\Rightarrow \theta^c = [\mathrm{I} - (F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}R]\,\hat{\theta} + [\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}R]\,\theta$$

$$= [\mathrm{I} - (F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[\,R\,\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}\,R\,](F^T\Psi^{-1}F)^{-1}\big(F^T\Psi^{-1}(F\theta + \omega)\big) + [\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}R]\theta$$

$$= [\mathrm{I} - (F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}\,R\,](F^T\Psi^{-1}F)^{-1}(F^T\Psi^{-1}F)\theta + [\mathrm{I} - (F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}\,R\,](F^T\Psi^{-1}F)^{-1}(F^T\Psi^{-1}\omega) + [\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}R]\,\theta$$

$$= \theta - (F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}\,R\theta + [\,\mathrm{I}\, - (F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}\,R\,](F^T\Psi^{-1}F)^{-1}(F^T\Psi^{-1}\omega) + (F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}R\,\theta$$

$$\Rightarrow\ \theta^c = \ \theta + \left[I - \lim_{NT\to\infty}\left(\frac{1}{NT}F^T\Psi^{-1}F + \frac{1}{NT}\Sigma^{-1}\right)^{-1}R^T\left[R\lim_{NT\to\infty}\left(\frac{1}{NT}F^T\Psi^{-1}F + \frac{1}{NT}\Sigma^{-1}\right)^{-1}R^T\right]^{-1}R\right]$$

$$\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}\left(\frac{1}{NT}F^T\Psi^{-1}\omega\right)$$

$$\Rightarrow \theta^c-\theta=\left[I-\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T\left[R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T\right]^{-1}R\right]$$

$$\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}\left(\frac{1}{NT}F^T\Psi^{-1}\omega\right)$$

$$\Rightarrow \theta^c-\theta=\tau\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}\left(\frac{1}{NT}F^T\Psi^{-1}\omega\right)$$

$$\therefore \sqrt{NT}(\theta^c-\theta)=\tau\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}\left(\frac{1}{\sqrt{NT}}F^T\Psi^{-1}\omega\right).$$

Now, the mean and covariance of $\sqrt{NT}\,(\theta^c-\theta)$ as following

$E[\sqrt{NT}\,(\theta^c-\theta)]=0$ (since $\theta^c=\theta$), and

$$cov[\sqrt{NT}\,(\theta^c-\theta)]=E[\sqrt{NT}\,(\theta^c-\theta)\sqrt{NT}\,(\theta^c-\theta)^T=E\left\{\tau\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}\left(\frac{1}{NT}F^T\Psi^{-1}\omega\omega^T\Psi^{-1}F\right)\right.$$

$$\left.\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}\tau^T\right\}=\left\{\tau\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}\frac{1}{NT}F^T\Psi^{-1}E(\omega\omega^T)\Psi^{-1}F\right)$$

$$\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}=\tau\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}\left(\frac{1}{NT}F^T\Psi^{-1}F\right)\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}\tau^T$$

$$=\tau\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}\tau^T$$

$$=[I-\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T\left[R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T\right]^{-1}R]$$

$$\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}[I-R^T\left[R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T\right]^{-1}R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}]$$

$$=[\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}-\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}$$

$$R^T\left[R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T\right]^{-1}R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}]$$

$$[I-R^T\left[R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T\right]^{-1}R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}]$$

$$=\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}-\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T\left[R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T\right]^{-1}$$

$$R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}-\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}$$

$$R^T\left[R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T\right]^{-1}R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}$$

$$+\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T\left[R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T\right]^{-1}R^T$$

$$R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}\left[R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T\right]^{-1}R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}$$

$$\Rightarrow cov\,[\sqrt{NT}\,(\theta^c-\theta)]=[\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}-\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T\left[R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T\right]^{-1}$$

$$R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}=\xi$$

$$\therefore \sqrt{NT}\,(\theta^c-\theta)\xrightarrow{d}N\,[0,\ \xi].\ ■$$

5 Simulation results

To demonstrate the efficacy of our methods, we produce observations from the model 1. The number of sections is 3, 5, 6, and 8, and the sample sizes are $n=100, 200$, and 300.

We consider three restrictions on model parameters as follows

$$\beta_1+\beta_2+2\beta_3=10$$
$$3\beta_1+\beta_3=2\beta_2+\beta_4$$
$$\beta_2+2\beta_4+\beta_5=12$$

Average mean squared error ($AMSE$) as well as average mean absolute error ($AMAE$) computations are used to measure the predicted models' goodness of fit. The (AMSE) and ($AMAE$) for the Bayesian and restricted Bayesian models are presented in Table (1) as summary values. This table demonstrates that for estimating the parameters of a study's model, the restricted Bayesian method outperforms the Bayesian method. The Bayesian random effects model's lower values for the ($AMSE$) and ($AMAE$) were (0.0020221241) and (0.0021100122), respectively, at a sample size of ($n=200$) and a number of sections of (5). While the lowest value of ($AMSE$) and ($AMAE$) for the constrained

Bayesian model were at the sample size of ($n = 300$) and the number of sections was (6). The values were respectively (0.0002052268) and (0.0002044004). Table (1) clearly shows that, for all sample sizes and number of selected sections, the estimate results of the restricted Bayesian model are much superior than the estimation results of the random-effects Bayesian model, as indicated by the values of the ($AMSE$) and ($AMAE$).

Table 1. Outcomes of the (***AMSE***) and (***AMAE***) Bayesian evaluation criteria and restricted Bayesian models

Models	Sample size	Number of Sections	(AMSE)	(AMAE)
Bayesian model	100	3	0.0046328124	0.0037434233
		5	0.0047325742	0.0037675632
		6	0.0047732752	0.0038425943
		8	0.0033427182	0.0032166464
	200	3	0.0031417255	0.0031275352
		5	0.0020221241	0.0021100122
		6	0.0036141322	0.0033715243
		8	0.0022211323	0.0026342226
	300	3	0.0030977601	0.0040101122
		5	0.0038338665	0.0033487501
		6	0.0033395522	0.0039995521
		8	0.0038222113	0.0040011221
Restricted Bayesian model	100	3	0.0020110622	0.0022110553
		5	0.0026411241	0.0030011321
		6	0.0030471371	0.0031131665
		8	0.0026607444	0.0022200013
	200	3	0.0030032101	0.0020131321
		5	0.0020190944	0.0020100334
		6	0.0020311457	0.0022011022
		8	0.0020098855	0.0021021347
	300	3	0.0002220221	0.0002200111
		5	0.0002110011	0.0002100333
		6	0.0002052268	0.0002044004
		8	0.0002300611	0.0002400001

6 Conclusion

1. The conjugate Bayesian estimator of the parameters vector (θ) of random effects model is

$$\breve{\theta} = (F^T\Psi^{-1}F + \Sigma^{-1})^{-1}(F^T\Psi^{-1}Y + \Sigma^{-1}\mu_\theta).$$

2. The Bayesian estimator of θ with linear constraints is

$$\theta^c = \breve{\theta} + (F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T[R\,(F^T\Psi^{-1}F + \Sigma^{-1})^{-1}R^T]^{-1}(r - R\breve{\theta}).$$

3. The restricted Bayesian estimator θ^c is weakly consistent for θ.

4. The restricted Bayesian estimator θ^c is asymptotically normally distributed with mean zero and covariance matrix ξ where

$$\xi=\{\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1} - \left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T[R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}R^T]R\left(\frac{1}{NT}F^T\Psi^{-1}F\right)^{-1}\}.$$

5. The lower values of the ($AMSE$) and ($AMAE$) for the Bayesian random effects model were at a sample size of ($n = 200$) and a number of sections of (5), where their values were (0.0020221241) and (0.0021100122) respectively. While the lowest value of ($AMSE$) and ($AMAE$) for the constrained Bayesian model were at the sample size of ($n = 300$) and the number of sections was (6). The values were respectively (0.0002052268) and (0.0002044004).

6. According to the values of the ($AMSE$) and ($AMAE$), for all sizes of the samples tested and for all numbers of selected sections, the estimation results of the restricted Bayesian model are significantly superior than the estimation results of the random-effects Bayesian model.

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