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Solving first order fuzzy partial differential equations using fuzzy Aboodh transform

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Abstract. In this article, we extend the concept of the fuzzy Aboodh transform to the solution of fuzzy partial differential equations using Hukuhara differentiability. In order to create the solution to first-order fuzzy partial differential equations, we first provide the fuzzy Aboodh transform and then present the basic definitions and theorems for the fuzzy Aboodh transform of fuzzy partial derivatives. Finally, the method is illustrated with an example to show the ability of the proposed method.

1 Introduction

Mathematical equations known as partial differential equations deal with several variables and their partial derivatives. When addressing issues in the actual world. They are better than regular differential equations. This is due to the fact that while witnessing events, several factors are frequently present at once. For instance. We must deal with the variables of distance and time simultaneously while modeling the heat transfer of a wire. The issues surrounding the topic have been extensively explored and researched by several scholars [1,2]. Partial differential equations have been used in various fields including biology, physics and engineering as can be observed in the literature [3,4]. Both analytical and numerical approaches have been suggested with a number of tools for solving partial differential equations [5,6]. This suggests that partial differential equations are essential for issue modeling. Partial differential equations are not always the ideal choice when addressing problems that occur in the actual world though. For instance. The model may not always be accurate when used to simulate specific dynamic phenomena using partial differential equations. This is a result of our poor understanding of the dynamic system. For instance. There can be fuzziness in the initial value. By combining partial differential equations and fuzzy set theory [7]. Buckley and Feuring [8] created fuzzy partial differential equations to address this issue. Allahviranloo [9]. Who presented the solution of fuzzy partial differential equations by difference approach utilizing Taylor series continued the work after that. Later. Pownuk [10] introduced a technique based on the sensitivity analysis and finite element approach to arrive at the solution of fuzzy partial differential equations. Since that time. other researchers have focused on studying fuzzy partial differential equations [11,12]. In a recent study by Bertone et al [13]. The authors used the Zadeh extension principle to examine fuzzy

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heat and fuzzy wave equations. Integral transformations are frequently extremely helpful in the solution of partial differential equations. This is true because integral transforms change the difficult original function into a new easier-to-solve function. We intend to use fuzzy Aboodh transform in many unresolved problems that involve several other fields, especially numerical analysis, statistics, computer science, and electronic engineering [14,15].

2 Preliminaries

This section recalls a number of definitions and characteristics of fuzzy numbers and fuzzy functions. The symbols $\mathcal{R}$ and $\mathcal{R}_{\mathcal{F}}$, respectively, stand for the real number and the fuzzy number.

Definition (2.1)[16]

The function $₣ : \mathcal{R} \to [0,1]$ is a fuzzy number if it satisfies:

1. $₣$ is upper semi-continuous.
2. $₣$ is fuzzy convex, i. e. $₣(\zeta x_1 + (1-\zeta)x_2) \geq \min\{₣(x_1), ₣(x_2)\}$, for all $x_1, x_2 \in \mathcal{R}$ and $\zeta \in [0,1]$.
3. $₣$ is normal i. e. $\exists\, x_0 \in \mathcal{R}$ for which $₣(x) = 1$.
4. Supp ($₣$) = $\{ x \in \mathcal{R} ; ₣(x) > 0 \}$, and cl (Supp($₣$)) (Closure Support) is a compact.

Definition (2.2) [17]

Ordered pair is a fuzzy number $(\underline{\varrho}, \overline{\varrho})$ of functions $\overline{\varrho}(\zeta)$, $\underline{\varrho}(\zeta)$.

$\zeta \in [0,1]$,which satisfies:

1. $\underline{\varrho}(\zeta)$ is a bounded non-decreasing left continuous function in (0, 1], and right continuous at 0.
2. $\overline{\varrho}(\zeta)$ is a bounded non- increasing left continuous function in (0, 1], and right continuous at 0.
3. $\underline{\varrho}(\zeta) \leq \overline{\varrho}(\zeta)$, $\zeta \in [0,1]$.

Definition (2.3) [17]

We recall that for $x_1 < x_2 < x_3$ which $x_1, x_2, x_3 \in \mathcal{R}$, triangular fuzzy number $ƒ = (x_1, x_2, x_3)$ determined by x_1, x_2, x_3 is given such that $\underline{\Gamma}(\zeta) = (x_2 - x_1)\zeta + x_1$ and $\overline{\Gamma}(\zeta) = -(x_3 - x_2)\zeta + x_3$ are the endpoints of the ζ -level sets, for all $\zeta \in [0,1]$.

For arbitrary $\Gamma = (\underline{\Gamma}(\zeta), \overline{\Gamma}(\zeta)), \Phi = (\underline{\Phi}(\zeta), \overline{\Phi}(\zeta)), 0 \leq \zeta \leq 1$ we define:

1. Addition $\Gamma \oplus \Phi = (\underline{\Gamma}(\zeta) + \underline{\Phi}(\zeta), \overline{\Gamma}(\zeta) + \overline{\Phi}(\zeta))$,
2. Scalar multiplication $\alpha \odot \Gamma = \begin{cases} (\alpha\underline{\Gamma}, \alpha\overline{\Gamma}) & \alpha \geq 0, \\ (\alpha\overline{\Gamma}, \alpha\underline{\Gamma}) & \alpha < 0, \end{cases}$

Definition (2.4) [18]

Assume that $\Gamma, \Phi \in \mathcal{R}_{\mathcal{F}}$,There is $\gamma \in \mathcal{R}_{\mathcal{F}}$ such that $\Gamma = \Phi + \gamma$ then Γ is known the H-differences of Γ and Φ and it is represented by $\Gamma \ominus \Phi$, Where $\Gamma \ominus \Phi = \gamma \Leftrightarrow \Gamma = \Phi + \gamma$.

Definition (2.5) [19]

Let Γ and Φ are fuzzy numbers then the distance between fuzzy numbers in the Hausdorff metric, which is given by, $\mathbb{D}: \mathcal{R}_{\mathcal{F}} \times \mathcal{R}_{\mathcal{F}} \to [0, +\infty)$, where $\mathcal{R}_{\mathcal{F}}$be the set of all fuzzy numbers on $\mathcal{R}$:

$$\mathbb{D}(\Gamma, \Phi) = \sup_{\zeta\in[0,1]} \max\{|\underline{\Gamma}(\zeta) - \underline{\Phi}(\zeta)|, |\overline{\Gamma}(\zeta) - \overline{\Phi}(\zeta)|\},$$

Where $\Gamma = \left(\underline{\Gamma}(\zeta), \overline{\Gamma}(\zeta)\right), \Phi = \left(\underline{\Phi}(\zeta), \overline{\Phi}(\zeta)\right) \subset \mathcal{R}$ and following properties are well known:

1. $\mathbb{D}(\Gamma \oplus \gamma, \Phi \oplus \gamma) = \mathbb{D}(\Gamma, \Phi), \forall\, \Gamma, \Phi, \gamma \in \mathcal{R}_{\mathcal{F}}$.
2. $\mathbb{D}(\zeta \odot \Gamma, \zeta \odot \Phi) = |\zeta|\, \mathbb{D}(\Gamma, \Phi), \forall\, \Gamma, \Phi \in \mathcal{R}_{\mathcal{F}}, \zeta \in \mathcal{R}$.
3. $\mathbb{D}(\Gamma \oplus \Gamma, \Phi \oplus \gamma) \leq \mathbb{D}(\Gamma, \Phi) + \mathbb{D}(\Gamma, \gamma), \forall\, \Gamma, \Phi, \gamma \in \mathcal{R}_{\mathcal{F}}$.

4. $(\mathbb{D}, \mathcal{R}_{\mathcal{F}})$ is a complete metric space.

Theorem (2.6) [20]

Assume $₣(x̧)$ be a fuzzy valued function on $[a,\infty)$ and it is represented by $((\underline{₣}(x̧,\zeta), \overline{₣}(x̧,\zeta))$, For any fixed $\zeta \in [0,1]$, let $\underline{₣}(x̧,\zeta)$ and $\overline{₣}(x̧,\zeta)$ are Riemann-integrals on $[a,b]$, assume there are two positive constant $\underline{M}(\zeta)$ and $\overline{M}(\zeta)$ such that $\int_a^b \left|\underline{₣}(x̧,\zeta)\right| dx̧ \le \underline{M}(\zeta)$ and $\int_a^b \left|\overline{₣}(x̧,\zeta)\right| dx̧ \le \overline{M}(\zeta)$,Then $₣(x̧)$ is improper fuzzy Riemann-integrable on $[a,\infty)$ and the improper fuzzy Riemann- integrable is a fuzzy number. Furthermore, we have:

$$\int_a^\infty ₣(x̧)\,dx̧ = \left[\int_a^\infty (\underline{₣}(x̧,\zeta)dx̧, \int_a^\infty \overline{₣}(x̧,\zeta)dx̧\right],$$

Definition (2.7)

Fuzzy valued function $₣ : (a,b)\times(a,b) \longrightarrow \mathcal{R}_{\mathcal{F}}$ and, We say that $₣$ is H-differentiable of the first order at

$x̧_0 \in (a,b)$ with respect to $x̧$, if there exists an element $\frac{\partial ₣(x̧_0,ɏ)}{\partial x} \in \mathcal{R}_{\mathcal{F}}$, such that:

i. $\forall \quad Ʊ > 0$ that is a sufficiently small, there exist $\frac{\partial ₣(x̧_0+Ʊ,ɏ)}{\partial x̧} \ominus \frac{\partial ₣(x̧_0,ɏ)}{\partial x̧}, \frac{\partial ₣(x̧_0,ɏ)}{\partial x̧} \ominus \frac{\partial ₣(x̧_0-Ʊ,tɏ)}{\partial x̧}$,

where $\lim\limits_{Ʊ\to 0} \frac{\frac{\partial ₣(x̧_0+Ʊ,t)}{\partial x̧} \ominus \frac{\partial ₣(x̧_0,t)}{\partial x̧}}{Ʊ} = \lim\limits_{Ʊ\to 0} \frac{\frac{\partial ₣(x̧_0,t)}{\partial x̧} \ominus \frac{\partial ₣(x̧_0-Ʊ,t)}{\partial x̧}}{Ʊ} = \frac{\partial ₣(x̧_0,t)}{\partial x̧}$

or

ii. $\forall \quad Ʊ > 0$ that is a sufficiently small, there exist $\frac{\partial ₣(x̧_0,ɏ)}{\partial x} \ominus \frac{\partial ₣(x̧_0+Ʊ,ɏ)}{\partial x̧}, \frac{\partial ₣(x̧_0-Ʊ,ɏ)}{\partial x̧} \ominus \frac{\partial ₣(x̧_0,ɏ)}{\partial x̧}$

where $\lim\limits_{Ʊ\to 0} \frac{\frac{\partial ₣(x̧_0,ɏ)}{\partial x̧} \ominus \frac{\partial ₣(x̧_0+Ʊ,ɏ)}{\partial x̧}}{-Ʊ} = \lim\limits_{Ʊ\to 0} \frac{\frac{\partial ₣(x̧_0-Ʊ,ɏ)}{\partial x̧} \ominus \frac{\partial ₣(x̧_0,ɏ)}{\partial x̧}}{-Ʊ} = \frac{\partial ₣(x̧_0,ɏ)}{\partial x̧}$,

Definition (2.8)

Fuzzy valued function $₣ : (a,b)\times(a,b) \longrightarrow \mathcal{R}_{\mathcal{F}}$ and, We say that $₣$ is H- differentiable of the first order at

$ɏ_0 \in (a,b)$ with respect to t, if there exists an element $\frac{\partial ₣(x̧,ɏ_0)}{\partial ɏ} \in \mathcal{R}_{\mathcal{F}}$, such that:

i. $\forall \quad Ʊ > 0$ that is a sufficiently small, there exist $\frac{\partial ₣(x̧,ɏ_0+Ʊ)}{\partial ɏ} \ominus \frac{\partial ₣(x̧,ɏ_0)}{\partial ɏ}, \frac{\partial ₣(x̧,ɏ_0)}{\partial ɏ} \ominus \frac{\partial ₣(x̧,ɏ_0-Ʊ)}{\partial ɏ}$,

where $\lim\limits_{Ʊ\to 0} \frac{\frac{\partial ₣(x̧,ɏ_0+Ʊ)}{\partial ɏ} \ominus \frac{\partial ₣(x̧,ɏ_0)}{\partial ɏ}}{Ʊ} = \lim\limits_{Ʊ\to 0} \frac{\frac{\partial ₣(x̧,ɏ_0)}{\partial ɏ} \ominus \frac{\partial ₣(x̧,ɏ_0-Ʊ)}{\partial ɏ}}{Ʊ} = \frac{\partial ₣(x̧,ɏ_0)}{\partial ɏ}$

or

ii. $\forall \quad Ʊ > 0$ that is a sufficiently small, there exist $\frac{\partial ₣(x̧,ɏ_0)}{\partial ɏ} \ominus \frac{\partial ₣(x̧,ɏ_0+Ʊ)}{\partial ɏ}, \frac{\partial ₣(x̧,ɏ_0-Ʊ)}{\partial ɏ} \ominus \frac{\partial ₣(x̧,ɏ_0)}{\partial ɏ}$

where $\lim\limits_{Ʊ\to 0} \frac{\frac{\partial ₣(x̧,ɏ_0)}{\partial ɏ} \ominus \frac{\partial ₣(x̧,ɏ_0+Ʊ)}{\partial ɏ}}{-Ʊ} = \lim\limits_{Ʊ\to 0} \frac{\frac{\partial ₣(x̧,ɏ_0-Ʊ)}{\partial ɏ} \ominus \frac{\partial ₣(x̧,ɏ_0)}{\partial ɏ}}{-Ʊ} = \frac{\partial ₣(x̧,ɏ_0)}{\partial ɏ}$,

Theorem (2.9)

Let $₣(x̧,ɏ): \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}_{\mathcal{F}}$ be a function and represents $₣(x̧,ɏ) = ((\underline{₣}(x̧,ɏ,\zeta), \overline{₣}(x̧,ɏ,\zeta))$ in each case for $\zeta \in [0,1]$, Then:

If $₣(x̧,ɏ)$ is a differentiable form i, then $(\underline{₣}(x̧,ɏ,\zeta)$ and $\overline{₣}(x̧,ɏ,\zeta)$ are differentiable functions and

$\frac{\partial F(x,y)}{\partial y} = (\frac{\partial \underline{F}(x,y,\zeta)}{\partial y}, \frac{\partial \overline{F}(x,y,\zeta)}{\partial y})$ or $\left(\frac{\partial \underline{F}(x,y)}{\partial y}, \frac{\partial \overline{F}(x,y)}{\partial y}\right) = \left(\frac{\partial \underline{F}(x,y,\zeta)}{\partial y}, \frac{\partial \overline{F}(x,y,\zeta)}{\partial y}\right)$.

If $F(x,y)$ is a differentiable form ii, then ($\underline{F}(x,y,\zeta)$ and $\overline{F}(x,y,\zeta)$ are differentiable functions and

$\frac{\partial F(x,y)}{\partial y} = (\frac{\partial \overline{F}(x,y,\zeta)}{\partial y}, \frac{\partial \underline{F}(x,y,\zeta)}{\partial y})$ or $(\frac{\partial \overline{F}(x,y)}{\partial y}, \frac{\partial \underline{F}(x,y)}{\partial y}) = (\frac{\partial \underline{F}(x,y,\zeta)}{\partial y}, \frac{\partial \overline{F}(x,y,\zeta)}{\partial y})$

Proof: (1)

Since $F(x,y)$ is a differentiable form i

$$\begin{cases} \frac{\partial F(x,y_0+\mho)}{\partial y} \ominus \frac{\partial F(x,y_0)}{\partial y} = \left[\frac{\partial \underline{F}(x,y_0+\mho)}{\partial y} - \frac{\partial \underline{F}(x,y_0)}{\partial y}, \frac{\partial \overline{F}(x,y_0+\mho)}{\partial y} - \frac{\partial \overline{F}(x,y_0)}{\partial y}\right], \\ \frac{\partial F(x,y_0)}{\partial y} \ominus \frac{\partial F(x,y_0-\mho)}{\partial y} = \left[\frac{\partial \underline{F}(x,y_0)}{\partial y} - \frac{\partial \underline{F}(x,y_0-\mho)}{\partial y}, \frac{\partial \overline{F}(x,y_0)}{\partial y} - \frac{\partial \overline{F}(x,y_0-\mho)}{\partial y}\right], \end{cases}$$

And multiplying by $\frac{1}{\mho}$, $\mho > 0$

$$\begin{cases} \frac{\frac{\partial F(x,y_0+\mho)}{\partial y} \ominus \frac{\partial F(x,y_0)}{\partial y}}{\mho} = \left[\frac{\frac{\partial \underline{F}(x,y_0+\mho)}{\partial y} - \frac{\partial \underline{F}(x,y_0)}{\partial y}}{\mho}, \frac{\frac{\partial \overline{F}(x,y_0+\mho)}{\partial y} - \frac{\partial \overline{F}(x,y_0)}{\partial y}}{\mho}\right], \\ \frac{\frac{\partial F(x,y_0)}{\partial y} \ominus \frac{\partial F(x,y_0-\mho)}{\partial y}}{\mho} = \left[\frac{\frac{\partial \underline{F}(x,y_0)}{\partial y} - \frac{\partial \underline{F}(x,y_0-\mho)}{\partial y}}{\mho}, \frac{\frac{\partial \overline{F}(x,y_0)}{\partial y} - \frac{\partial \overline{F}(x,y_0-\mho)}{\partial y}}{\mho}\right], \end{cases}$$

Therefore

$$\begin{cases} \lim_{\mho \to 0} \frac{\frac{\partial F(x,y_0+\mho)}{\partial y} \ominus \frac{\partial F(x,y_0)}{\partial y}}{\mho} = \lim_{\mho \to 0} \left[\frac{\frac{\partial \underline{F}(x,y_0+\mho)}{\partial y} - \frac{\partial \underline{F}(x,y_0)}{\partial y}}{\mho}, \frac{\frac{\partial \overline{F}(x,y_0+\mho)}{\partial y} - \frac{\partial \overline{F}(x,y_0)}{\partial y}}{\mho}\right], \\ \lim_{\mho \to 0} \frac{\frac{\partial F(x,y_0)}{\partial y} \ominus \frac{\partial F(x,y_0-\mho)}{\partial y}}{\mho} = \lim_{\mho \to 0} \left[\frac{\frac{\partial \underline{F}(x,y_0)}{\partial y} - \frac{\partial \underline{F}(x,y_0-\mho)}{\partial y}}{\mho}, \frac{\frac{\partial \overline{F}(x,y_0)}{\partial y} - \frac{\partial \overline{F}(x,y_0-\mho)}{\partial y}}{\mho}\right], \end{cases}$$

Then $\frac{\partial F(x,y)}{\partial y} = \left(\frac{\partial \underline{F}(x,y,\zeta)}{\partial y}, \frac{\partial \overline{F}(x,y,\zeta)}{\partial y}\right)$.

Proof: (2)

Since $F(x,y)$ is a differentiable form ii

$$\begin{cases} \frac{\partial F(x,y_0)}{\partial y} \ominus \frac{\partial F(x,y_0+\mho)}{\partial y} = \left[\frac{\partial \underline{F}(x,y_0)}{\partial y} - \frac{\partial \underline{F}(x,y_0+\mho)}{\partial y}, \frac{\partial \overline{F}(x,y_0)}{\partial y} - \frac{\partial \overline{F}(x,y_0+\mho)}{\partial y}\right], \\ \frac{\partial F(x,y_0-\mho)}{\partial y} \ominus \frac{\partial F(x,y_0)}{\partial y} = \left[\frac{\partial \underline{F}(x,y_0-\mho)}{\partial y} - \frac{\partial \underline{F}(x,y_0)}{\partial y}, \frac{\partial \overline{F}(x,y_0-\mho)}{\partial y} - \frac{\partial \overline{F}(x,y_0)}{\partial y}\right], \end{cases}$$

And multiplying by $\frac{1}{-\mho}$, $\mho > 0$

$$\begin{cases} \frac{\frac{\partial F(x,y_0)}{\partial y} \ominus \frac{\partial F(x,y_0+\mho)}{\partial y}}{-\mho} = \left[\frac{\frac{\partial \underline{F}(x,y_0)}{\partial y} - \frac{\partial \underline{F}(x,y_0+\mho)}{\partial y}}{-\mho}, \frac{\frac{\partial \overline{F}(x,y_0)}{\partial y} - \frac{\partial \overline{F}(x,y_0+\mho)}{\partial y}}{-\mho}\right], \\ \frac{\frac{\partial F(x,y_0-\mho)}{\partial y} \ominus \frac{\partial F(x,y_0)}{\partial y}}{-\mho} = \left[\frac{\frac{\partial \underline{F}(x,y_0-\mho)}{\partial y} - \frac{\partial \underline{F}(x,y_0)}{\partial y}}{-\mho}, \frac{\frac{\partial \overline{F}(x,y_0-\mho)}{\partial y} - \frac{\partial \overline{F}(x,y_0)}{\partial y}}{-\mho}\right], \end{cases}$$

With simple calculation:

$$\begin{cases} \dfrac{\dfrac{\partial Ꞙ(x, y_0)}{\partial y} \ominus \dfrac{\partial Ꞙ(x, y_0 + \mho)}{\partial y}}{-\mho} = \left[\dfrac{\dfrac{\partial \overline{Ꞙ}(x, y_0 + \mho)}{\partial y} - \dfrac{\partial \overline{Ꞙ}(x, y_0)}{\partial y}}{\mho}, \dfrac{\dfrac{\partial \underline{Ꞙ}(x, y_0 + \mho)}{\partial y} - \dfrac{\partial \underline{Ꞙ}(x, y_0)}{\partial y}}{\mho}\right], \\ \dfrac{\dfrac{\partial Ꞙ(x, y_0 - \mho)}{\partial y} \ominus \dfrac{\partial Ꞙ(x, y_0)}{\partial y}}{-\mho} = \left[\dfrac{\dfrac{\partial \overline{Ꞙ}(x, y_0)}{\partial y} - \dfrac{\partial \overline{Ꞙ}(x, y_0 - \mho)}{\partial y}}{\mho}, \dfrac{\dfrac{\partial \underline{Ꞙ}(x, y_0)}{\partial y} - \dfrac{\partial \underline{Ꞙ}(x, y_0 - \mho)}{\partial y}}{\mho}\right], \end{cases}$$

Therefore

$$\begin{cases} \lim_{\mho \to 0} \dfrac{\dfrac{\partial Ꞙ(x, y_0)}{\partial y} \ominus \dfrac{\partial Ꞙ(x, y_0 + \mho)}{\partial y}}{-\mho} = \lim_{\mho \to 0} \left[\dfrac{\dfrac{\partial \overline{Ꞙ}(x, y_0 + \mho)}{\partial y} - \dfrac{\partial \overline{Ꞙ}(x, y_0)}{\partial y}}{\mho}, \dfrac{\dfrac{\partial \underline{Ꞙ}(x, y_0 + \mho)}{\partial y} - \dfrac{\partial \underline{Ꞙ}(x, y_0)}{\partial y}}{\mho}\right], \\ \lim_{\mho \to 0} \dfrac{\dfrac{\partial Ꞙ(x, y_0 - \mho)}{\partial y} \ominus \dfrac{\partial Ꞙ(x, y_0)}{\partial y}}{-\mho} = \lim_{\mho \to 0} \left[\dfrac{\dfrac{\partial \overline{Ꞙ}(x, y_0)}{\partial y} - \dfrac{\partial \overline{Ꞙ}(x, y_0 - \mho)}{\partial y}}{\mho}, \dfrac{\dfrac{\partial \underline{Ꞙ}(x, y_0)}{\partial y} - \dfrac{\partial \underline{Ꞙ}(x, y_0 - \mho)}{\partial y}}{\mho}\right], \end{cases}$$

Then $\frac{\partial Ꞙ(x,y)}{\partial y} = \left(\frac{\partial \overline{Ꞙ}(x,y,\zeta)}{\partial y}, \frac{\partial \underline{Ꞙ}(x,y,\zeta)}{\partial y}\right)$.

3 The fuzzy Aboodh transform for fuzzy partial derivatives

There is no possibility to apply the fuzzy Aboodh transform [21- 23] directly to solve fuzzy partial differential equations, so we relied on Hukuhara to generate all possible cases and, according to the order, solve the equation using the fuzzy Aboodh transform. Then two novel fuzzy partial derivative fuzzy Aboodh transform findings are presented.

Definition (3.1)

Let $Ꞙ(x)$ be a continuous fuzzy-valued function, Assume that $\frac{1}{s} Ꞙ(x)e^{-sx}$ is an improper fuzzy Rimann-integrable on $[0, \infty)$, then $\frac{1}{s}\int_0^\infty Ꞙ(x)e^{-sx}\, dx$ is being called fuzzy Aboodh transform and is referred to as

$$\widehat{A}[Ꞙ(x)] = \frac{1}{s}\int_0^\infty Ꞙ(x)e^{-sx}\, dx, \ (s > 0 \text{ and integer })$$

$$\frac{1}{s}\int_0^\infty Ꞙ(x)e^{-sx}\, dx = \left(\frac{1}{s}\int_0^\infty \underline{Ꞙ}(x, \zeta)e^{-sx}\, dx, \frac{1}{s}\int_0^\infty \overline{Ꞙ}(x, \zeta)e^{-sx}\, dx\right),$$

Fuzzy Aboodh transform can be rewritten into parametric form as follow,

$$\widehat{A}[Ꞙ(x)] = \left(A[\underline{Ꞙ}(x, \zeta)], A[\overline{Ꞙ}(x, \zeta)]\right),$$

In the following, we introduce new results of fuzzy Aboodh transform for fuzzy partial derivatives,

Definition (3.2)

Let $Ꞙ(x, y): [0, \infty) \times [0, \infty) \longrightarrow \mathcal{R}_{\mathcal{F}}$ be a continuous fuzzy-valued function where $y \geq 0$ is a time variable, Denote by $\widehat{A}[Ꞙ(x, y)]$ the fuzzy Aboodh transform of Ꞙ with respect to y, that is to say

$$\widehat{A}[Ꞙ(x, s)] = \widehat{A}\{Ꞙ(x, y)\} = \frac{1}{s}\int_0^\infty Ꞙ(x, y)e^{-sy}\, dy$$

Indeed, we can present the above definition for fuzzy Aboodh transform based on the ζ -cut representation of fuzzy-valued function Ꞙ as following:

$$\widehat{A}[Ꞙ(x, s)] = \widehat{A}\{Ꞙ(x, y)\} = \left(A[\underline{Ꞙ}(x, y, \zeta)], A[\overline{Ꞙ}(x, y, \zeta)]\right), 0 \leq \zeta \geq 1,$$

Where, $A\left[\underline{Ғ}(x, y, \zeta)\right]=\frac{1}{s}\int_0^\infty \underline{Ғ}(x, y, \zeta)e^{-sy}\,dy$ and $A\left[\overline{Ғ}(x, y, \zeta)\right]=\frac{1}{s}\int_0^\infty \overline{Ғ}(x, y, \zeta)e^{-sy}\,dy$.

Theorem (3.3)

Let $Ғ(x, y)\colon[0,\infty)\times[0,\infty)\longrightarrow\mathcal{R}_{\mathcal{F}}$ be a continuous fuzzy-valued function, Assume that $\frac{1}{s}Ғ(x, y)\,e^{-sx}$ is an improper fuzzy Rimann-integrable on $[0,\infty)$. Then

$$\hat{A}\left[\frac{\partial Ғ(x, y)}{\partial x}\right]=\frac{\partial}{\partial x}\hat{A}[Ғ(x, s)]=\frac{d}{dx}\hat{A}[Ғ(x, s)]$$

Proof

$$\hat{A}\left[\frac{\partial Ғ(x, y)}{\partial x}\right]=\frac{1}{s}\int_0^\infty\frac{\partial}{\partial x}Ғ(x, y)e^{-sy}\,dy=\left[\frac{1}{s}\int_0^\infty\frac{\partial}{\partial x}\underline{Ғ}(x, y)e^{-sy}\,dy,\frac{1}{s}\int_0^\infty\frac{\partial}{\partial x}\overline{Ғ}(x, y)e^{-sy}\,dy\right]$$

$$\hat{A}\left[\frac{\partial Ғ(x, y)}{\partial x}\right]=\left[\frac{\partial}{\partial x}\frac{1}{s}\int_0^\infty\underline{Ғ}(x, y)e^{-sy}\,dy,\frac{\partial}{\partial x}\frac{1}{s}\int_0^\infty\overline{Ғ}(x, y)e^{-sy}\,dy\right]$$

$$\hat{A}\left[\frac{\partial Ғ(x, y)}{\partial x}\right]=\frac{\partial}{\partial x}\left[\frac{1}{s}\int_0^\infty\underline{Ғ}(x, y)e^{-sy}\,dy,\frac{1}{s}\int_0^\infty\overline{Ғ}(x, y)e^{-sy}\,dy\right]$$

$$\hat{A}\left[\frac{\partial Ғ(x, y)}{\partial x}\right]=\frac{\partial}{\partial x}\left[\hat{A}[Ғ(x, s)]\right]=\frac{d}{dx}\hat{A}[Ғ(x, s)]$$

Theorem (3.4)

Let $Ғ(x, y)$ be a primitive of $\frac{\partial Ғ(x, y)}{\partial y}$ on $[0,\infty)$ and $Ғ(x, y)$ be an integrable fuzzy-valued function then:

a. $Ғ(x, y)$ is (i)-differentiable then $\hat{A}\left[\frac{\partial Ғ(x, y)}{\partial y}\right]=s\hat{A}[Ғ(x, s)]\ominus\frac{1}{s}Ғ(x, 0)$.

b. $Ғ(x, y)$ is (ii)-differentiable then $\hat{A}\left[\frac{\partial Ғ(x, y)}{\partial y}\right]=-\frac{1}{s}Ғ(x, 0)\ominus -s\hat{A}[Ғ(x, s)]$.

Proof (a)

For an arbitrary fixed $\zeta\in[0,1]$,

$$s\hat{A}[Ғ(x, s)]\ominus\frac{1}{s}Ғ(x, 0)=\left(sA\left[\underline{Ғ}(x, s)\right]-\frac{1}{s}\underline{Ғ}(x, 0),sA\left[\overline{Ғ}(x, s)\right]-\frac{1}{s}\overline{Ғ}(x, 0)\right)$$

Since, $A\left[\frac{\partial\underline{Ғ}(x, y)}{\partial y}\right]=sA\left[\underline{Ғ}(x, s)\right]-\frac{1}{s}\underline{Ғ}(x, 0)$ and $A\left[\frac{\partial\overline{Ғ}(x, y)}{\partial y}\right]=sA\left[\overline{Ғ}(x, s)\right]-\frac{1}{s}\overline{Ғ}(x, 0)$,

Since $Ғ(x, y)$ is differentiable of form i using Theorem (2,9):

$$\frac{\underline{\partial Ғ}(x, y)}{\partial y}=\frac{\partial\underline{Ғ}(x, y, \zeta)}{\partial y},\frac{\overline{\partial Ғ}(x, y)}{\partial y}=\frac{\partial\overline{Ғ}(x, y, \zeta)}{\partial y},$$

$$A\left[\frac{\underline{\partial Ғ}(x, y)}{\partial y}\right]=sA\left[\underline{Ғ}(x, s)\right]-\frac{1}{s}\underline{Ғ}(x, 0),$$

$$A\left[\frac{\overline{\partial Ғ}(x, y)}{\partial y}\right]=sA\left[\overline{Ғ}(x, s)\right]-\frac{1}{s}\overline{Ғ}(x, 0),$$

$$s\hat{A}[Ғ(x, s)]\ominus\frac{1}{s}Ғ(x, 0)=\left(A\left[\frac{\underline{\partial Ғ}(x, y)}{\partial y}\right],A\left[\frac{\overline{\partial Ғ}(x, y)}{\partial y}\right]\right)$$

$$\hat{A}\left[\frac{\partial}{\partial y}Ғ(x, y)\right]=s\hat{A}[Ғ(x, s)]\ominus\frac{1}{s}Ғ(x, 0),$$

(b)

$$-\frac{1}{s}Ғ(x, 0)\ominus -s\hat{A}[Ғ(x, s)]=\left(-\frac{1}{s}\overline{Ғ}(x, 0)+sA\left[\overline{Ғ}(x, s)\right],-\frac{1}{s}\underline{Ғ}(x, 0)+sA\left[\underline{Ғ}(x, s)\right]\right),$$

Since, $A\left[\frac{\partial \underline{F}(x,y)}{\partial y}\right] = sA[\underline{F}(x,s)] - \frac{1}{s}\underline{F}(x,0)$ and $A\left[\frac{\partial \overline{F}(x,y)}{\partial y}\right] = sA[\overline{F}(x,s)] - \frac{1}{s}\overline{F}(x,0)$,

Since $F(x,y)$ is differentiable of form ii using Theorem (2,9):

$$\frac{\partial \overline{F}(x,y)}{\partial y} = \frac{\partial \underline{F}(x,y,\zeta)}{\partial y}, \frac{\partial \underline{F}(x,y)}{\partial y} = \frac{\partial \overline{F}(x,y,\zeta)}{\partial y},$$

$$\frac{\partial \underline{F}(x,y)}{\partial y} = -\frac{1}{s}\overline{F}(x,0) + sA[\overline{F}(x,s)],$$

$$\frac{\partial \overline{F}(x,y)}{\partial y} = -\frac{1}{s}\underline{F}(x,0) + sA[\underline{F}(x,s)],$$

$$-\frac{1}{s}F(x,0) \ominus -s\widehat{A}[F(x,s)] = \left(\frac{\partial \underline{F}(x,y)}{\partial y}, \frac{\partial \overline{F}(x,y)}{\partial y}\right),$$

$$\widehat{A}\left[\frac{\partial F(x,y)}{\partial y}\right] = -\frac{1}{s}F(x,0) \ominus -s\widehat{A}[F(x,s)],$$

4 Fuzzy Aboodh transform for fuzzy partial differential equations

Consider the following Fuzzy Partial Differential Equations,

$$\begin{cases} \frac{\partial F(x,y,\zeta)}{\partial x} = \mathcal{C}\,\frac{\partial F(x,y,\zeta)}{\partial y} + \mathcal{F}(x,y,F(x,y),\zeta) \\ F(x,0,\zeta) = \mathcal{G}(x,\zeta) = [\underline{\mathcal{G}}(x,\zeta), \overline{\mathcal{G}}(x,\zeta)] \\ F(0,y,\zeta) = r(y,\zeta) = [\underline{r}(y,\zeta), \overline{r}(y,\zeta)] \end{cases} \quad (4.1)$$

Where $F(x,y): [0,\infty) \times [0,\infty) \longrightarrow \mathcal{R}_{\mathcal{F}}$ is fuzzy function where $x, y \geq 0$ is a time variable and $\mathcal{C}$ is a constant real number, Furthermore it $\mathcal{G}(x,\zeta)$ and $r(x,\zeta)$ are the initial values and $\mathcal{F}(x,y,F(x,y),\zeta)$ is a fuzzy-valued function, Using Fuzzy Aboodh Transform for the equation (4,1), The solutions can be divided into four cases.

If $F(x,y)$ is (i)-differentiable both x and y then

$$\begin{cases} A\left[\frac{\partial \underline{F}(x,y,\zeta)}{\partial x}\right] = A\left[\mathcal{C}\,\frac{\partial \underline{F}(x,y,\zeta)}{\partial y}\right] + A[\underline{\mathcal{F}}(x,y,F(x,y),\zeta)], \\ A\left[\frac{\partial \overline{F}(x,y,\zeta)}{\partial x}\right] = A\left[\mathcal{C}\,\frac{\partial \overline{F}(x,y,\zeta)}{\partial y}\right] + A[\overline{\mathcal{F}}(x,y,F(x,y),\zeta)] \end{cases} \quad (4.2)$$

Using Theorems (3.3) and (3.4) on both sides of the equation (4.2),

$$\begin{cases} \frac{\partial}{\partial x}A[\underline{F}(x,y,\zeta)] = \mathcal{C}\,sA[\underline{F}(x,y,\zeta)] - \mathcal{C}\,\frac{1}{s}\underline{F}(x,0,\zeta) + A[\underline{\mathcal{F}}(x,y,F(x,y),\zeta)], \\ \frac{\partial}{\partial x}A[\overline{F}(x,y,\zeta)] = \mathcal{C}\,sA[\overline{F}(x,y,\zeta)] - \mathcal{C}\,\frac{1}{s}\overline{F}(x,0,\zeta) + A[\overline{\mathcal{F}}(x,y,F(x,y),\zeta)] \end{cases} \quad (4.3)$$

Solving the equation (4.3), satisfying the initial condition

$$\begin{cases} A[\underline{F}(x,y,\zeta)] = \mathbb{P}_1(\zeta,s) \\ A[\overline{F}(x,y,\zeta)] = \mathbb{N}_1(\zeta,s) \end{cases} \quad (4.5)$$

Where $\mathbb{P}_1(\zeta,s)$ and $\mathbb{N}_1(\zeta,s)$ are solution of the equation (4.5). Using the inverse Aboodh transform,

$$\begin{cases} [\underline{F}(x,y,\zeta)] = A^{-1}\mathbb{P}_1(\zeta,s) \\ [\overline{F}(x,y,\zeta)] = A^{-1}\mathbb{N}_1(\zeta,s) \end{cases}$$

$$[\underline{F}(x,y,\zeta)] = A^{-1}\mathbb{P}_1(\zeta,s), [\overline{F}(x,y,\zeta)] = A^{-1}\mathbb{N}_1(\zeta,s)$$

If $F(x,y)$ is (i)-differentiable for x $F(x,y)$ is (ii)-differentiable for y, then

$$\begin{cases} A\left[\frac{\partial \underline{F}(x,y,\zeta)}{\partial x}\right] = A\left[\mathcal{C}\ \frac{\partial \overline{F}(x,y,\zeta)}{\partial y}\right] + A\left[\underline{\mathcal{F}}(x,y,F(x,y),\zeta)\right], \\ A\left[\frac{\partial \overline{F}(x,y,\zeta)}{\partial x}\right] = A\left[\mathcal{C}\ \frac{\partial \underline{F}(x,y,\zeta)}{\partial y}\right] + A\left[\overline{\mathcal{F}}(x,y,F(x,y),\zeta)\right] \end{cases} \tag{4.6}$$

Using Theorems (3.3) and (3.4) on both sides of the equation (4.6).

$$\begin{cases} \frac{\partial}{\partial x} A\left[\underline{F}(x,y,\zeta)\right] = \mathcal{C}\ sA\left[\overline{F}(x,y,\zeta)\right] - \mathcal{C}\ \frac{1}{s}\overline{F}(x,0,\zeta) + A\left[\underline{\mathcal{F}}(x,y,F(x,y),\zeta)\right], \\ \frac{\partial}{\partial x} A\left[\overline{F}(x,y,\zeta)\right] = \mathcal{C}\ sA\left[\underline{F}(x,y,\zeta)\right] - \mathcal{C}\ \frac{1}{s}\underline{F}(x,0,\zeta) + A\left[\overline{\mathcal{F}}(x,y,F(x,y),\zeta)\right] \end{cases} \tag{4.7}$$

Solving the equation (4.7), satisfying the initial condition

$$\begin{cases} A\left[\underline{F}(x,y,\zeta)\right] = \mathbb{P}_2(\zeta,s) \\ A\left[\overline{F}(x,y,\zeta)\right] = \mathbb{N}_2(\zeta,s) \end{cases} \tag{4.8}$$

Where $\mathbb{P}_2(\zeta,s)$ and $\mathbb{N}_2(\zeta,s)$ are solution of the equation (4.8), Using the inverse Aboodh transform,

$$\begin{cases} \left[\underline{F}(x,y,\zeta)\right] = A^{-1}\mathbb{P}_2(\zeta,s) \\ \left[\overline{F}(x,y,\zeta)\right] = A^{-1}\mathbb{N}_2(\zeta,s) \end{cases}$$

$$\left[\underline{F}(x,y,\zeta)\right] = A^{-1}\mathbb{P}_2(\zeta,s), \left[\overline{F}(x,y,\zeta)\right] = A^{-1}\mathbb{N}_2(\zeta,s)$$

If $F(x,y)$ is (ii)-differentiable for x $F(x,y)$ is (i)-differentiable for y, then

$$\begin{cases} A\left[\frac{\partial \overline{F}(x,y,\zeta)}{\partial x}\right] = A\left[\mathcal{C}\ \frac{\partial \underline{F}(x,y,\zeta)}{\partial y}\right] + A\left[\underline{\mathcal{F}}(x,y,F(x,y),\zeta)\right], \\ A\left[\frac{\partial \underline{F}(x,y,\zeta)}{\partial x}\right] = A\left[\mathcal{C}\ \frac{\partial \overline{F}(x,y,\zeta)}{\partial y}\right] + A\left[\overline{\mathcal{F}}(x,y,F(x,y),\zeta)\right] \end{cases} \tag{4.9}$$

Using Theorems (3.3) and (3.4) on both sides of the equation (4,9),

$$\begin{cases} \frac{\partial}{\partial x} A\left[\overline{F}(x,y,\zeta)\right] = \mathcal{C}\ sA\left[\underline{F}(x,y,\zeta)\right] - \mathcal{C}\ \frac{1}{s}\underline{F}(x,0,\zeta) + A\left[\underline{\mathcal{F}}(x,y,F(x,y),\zeta)\right], \\ \frac{\partial}{\partial x} A\left[\underline{F}(x,y,\zeta)\right] = \mathcal{C}\ sA\left[\overline{F}(x,y,\zeta)\right] - \mathcal{C}\ \frac{1}{s}\overline{F}(x,0,\zeta) + A\left[\overline{\mathcal{F}}(x,y,F(x,y),\zeta)\right] \end{cases} \tag{4.10}$$

Solving the equation (4.10), satisfying the initial condition

$$\begin{cases} A\left[\underline{F}(x,y,\zeta)\right] = \mathbb{P}_3(\zeta,s) \\ A\left[\overline{F}(x,y,\zeta)\right] = \mathbb{N}_3(\zeta,s) \end{cases} \tag{4.11}$$

Where $\mathbb{P}_3(\zeta,s)$ and $\mathbb{N}_3(\zeta,s)$ are solution of the equation (4.11), Using the inverse Aboodh transform,

$$\begin{cases} \left[\underline{F}(x,y,\zeta)\right] = A^{-1}\mathbb{P}_3(\zeta,s) \\ \left[\overline{F}(x,y,\zeta)\right] = A^{-1}\mathbb{N}_3(\zeta,s) \end{cases}$$

$$\left[\underline{F}(x,y,\zeta)\right] = A^{-1}\mathbb{P}_3(\zeta,s), \left[\overline{F}(x,y,\zeta)\right] = A^{-1}\mathbb{N}_3(\zeta,s)$$

If $F(x,y)$ is (ii)-differentiable both x and y then

$$\begin{cases} A\left[\frac{\partial \overline{F}(x,y,\zeta)}{\partial x}\right] = A\left[\mathcal{C}\ \frac{\partial \overline{F}(x,y,\zeta)}{\partial y}\right] + A\left[\underline{\mathcal{F}}(x,y,F(x,y),\zeta)\right], \\ A\left[\frac{\partial \underline{F}(x,y,\zeta)}{\partial x}\right] = A\left[\mathcal{C}\ \frac{\partial \underline{F}(x,y,\zeta)}{\partial y}\right] + A\left[\overline{\mathcal{F}}(x,y,F(x,y),\zeta)\right] \end{cases} \tag{4.12}$$

Using Theorems (3.3) and (3.4) on both sides of the equation (4,12),

$$\begin{cases} \frac{\partial}{\partial x} A\left[\overline{F}(x,y,\zeta)\right] = \mathcal{C}\ sA\left[\overline{F}(x,y,\zeta)\right] - \mathcal{C}\ \frac{1}{s}\overline{F}(x,0,\zeta) + A\left[\underline{\mathcal{F}}(x,y,F(x,y),\zeta)\right], \\ \frac{\partial}{\partial x} A\left[\underline{F}(x,y,\zeta)\right] = \mathcal{C}\ sA\left[\underline{F}(x,y,\zeta)\right] - \mathcal{C}\ \frac{1}{s}\underline{F}(x,0,\zeta) + A\left[\overline{\mathcal{F}}(x,y,F(x,y),\zeta)\right] \end{cases} \tag{4.13}$$

Solving the equation (4.13), satisfying the initial condition

$$\begin{cases} A\left[\underline{F}(x,y,\zeta)\right] = \mathbb{P}_4(\zeta,s) \\ A\left[\overline{F}(x,y,\zeta)\right] = \mathbb{N}_4(\zeta,s) \end{cases} \tag{4.14}$$

Where $\mathbb{P}_3(\zeta,s)$ and $\mathbb{N}_3(\zeta,s)$ are solution of the equation (4.14), Using the inverse Aboodh transform,

$$\begin{cases} [\underline{F}(x, y, \zeta)] = A^{-1}\mathbb{P}_4(\zeta, s) \\ [\overline{F}(x, y, \zeta)] = A^{-1}\mathbb{N}_4(\zeta, s) \end{cases}$$

$$[\underline{F}(x, y, \zeta)] = A^{-1}\mathbb{P}_4(\zeta, s), [\overline{F}(x, y, \zeta)] = A^{-1}\mathbb{N}_4(\zeta, s)$$

Example (4.1): Consider a fuzzy partial differential equations:

$$\frac{\partial F(x, y)}{\partial x} = 3\frac{\partial F(x, y)}{\partial y} + x, \quad F(x, 0) = 3x(\zeta - 1, 1 - \zeta) + \frac{x^2}{2}, F(0, y) = y(\zeta - 1, 1 - \zeta),$$

Solution:

Apply fuzzy Aboodh transform on both sides to get

$$\widehat{A}\left[\frac{\partial F(x, y)}{\partial x}\right] = \widehat{A}\left[3\frac{\partial F(x, y)}{\partial y} + x\right]$$

Case (1) If $F(x, y)$ is (i)-differentiable both x and y then

$$[\underline{F}(x, y, \zeta)] = \frac{x^2}{2} + 3x(\zeta - 1) + y(\zeta - 1)$$

$$[\overline{F}(x, y, \zeta)] = \frac{x^2}{2} + 3x(1 - \zeta) + y(1 - \zeta)$$

Case (2) If $F(x, y)$ is (i)-differentiable for x $F(x, y)$ is (ii)-differentiable for y, then

$$[\underline{F}(x, y, \zeta)] = \frac{x^2}{2} + 3x(\zeta - 1) - y(\zeta - 1) + 2(\zeta - 1)(y - 3x)\mathcal{H}(y - 3x)$$

$$[\overline{F}(x, y, \zeta)] = \frac{x^2}{2} + 3x(1 - \zeta) - y(1 - \zeta) + 2(1 - \zeta)(y - 3x)\mathcal{H}(y - 3x)$$

Case (3) If $F(x, y)$ is (ii)-differentiable for x $F(x, y)$ is (i)-differentiable for y, then

$$[\underline{F}(x, y, \zeta)] = \frac{x^2}{2} + 3x(1 - \zeta) - y(1 - \zeta) + 2(1 - \zeta)(y - 3x)\mathcal{H}(y - 3x)$$

$$[\overline{F}(x, y, \zeta)] = \frac{x^2}{2} + 3x(\zeta - 1) - y(\zeta - 1) + 2(\zeta - 1)(y - 3x)\mathcal{H}(y - 3x)$$

Case (4) If $F(x, y)$ is (ii)-differentiable both x and y

$$[\underline{F}(x, y, \zeta)] = \frac{x^2}{2} + 3x(1 - \zeta) + y(1 - \zeta)$$

$$[\overline{F}(x, y, \zeta)] = \frac{x^2}{2} + 3x(\zeta - 1) + y(\zeta - 1)$$

5 Conclusion

In this paper the fuzzy Aboodh transform has been investigated in order to solve fuzzy partial differential equations. This method is considered a starting point for applying the fuzzy Aboodh transform to solving higher-order fuzzy partial differential equations. Two brand-new fuzzy Aboodh transform theorems for fuzzy partial differential equations have also been proposed in addition to this. In order to solve fuzzy partial differential equations. A technique using the fuzzy Aboodh transform is devised. A numerical example has been used to demonstrate the use of the fuzzy Aboodh transform on fuzzy partial differential equations.

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