Fear induce bistability in an eco-epidemiological model involving prey refuge and hunting cooperation

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Abstract. This work included a prey-predator eco-epidemiological model where the predator exhibits hunting cooperation factor, and the infected prey exhibits refuge and fear factors. By merging these factors, we endeavor to prove a thorough comprehension of the underlying mechanisms that command the stability and dynamics of eco-epidemiological systems. Mathematically the positiveness and boundedness of system solution are established. Existence conditions of system equilibria and their stability are found and analyzed by testing the characteristic equations of the system. Also, numerical simulation is carried out and which showed bistability and different stability behavior of the system as results of the effect of the system's parameters.

1 Introduction

In our diverse world, the "predator-prey" relationship is one of many forms of interspecies interactions that is crucial in determining the dynamics of intricate ecological systems. The dynamic of eco-epidemiological models is one of the main theme in mathematical biology, it considers the dynamics of infectious disease spreading among the ecosystem, notably prey-predator models with infectious diseases. Many studies shown that prey-predator relations have a significant impact on the way the ecosystems are organized, even though they considering predation as the alone source of interaction between the predator and its prey [1-4]. Recently in some theoretical and experimental studies prey individuals have been noted to alter their typical foraging behavior due the psychological strain of being discovered and slain by predators [5-8]. Through these and other studies, researchers have shown how crucial indirect effects like fear are in determining the dynamics between preys and predators.

In the last few years, many prey-predator mathematical models have been developed to investigate the consequences of fear effect. In Ref. [9-12] ecological models with fear effect are investigated, while, fear effect in eco-epidemiological models are thoughtful in Ref. [13-17]. Various biological deductions have been reached based on the different assumptions included in these mathematical models. Researchers in [18-19] discovered that the impact of fear can minify the numbers of various species, both prey and predators.

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Additionally, this fear-induced effect may initiate a process where diseases within the prey or predator population have a tendency to go extinct.

In ecological systems, several predators species are living in social groups for a variety of reasons, the most crucial being mating, more effective hunting and team attacks on large prey or other predators, in addition to increased protection against other predators. In ecosystems the widespread and important of population cooperation is shown in many different taxa, including birds, aquatic organisms, and carnivores [20-22]. Through the use of mathematical, ecological, and eco-epidemiological models, lots of studies have been investigated the implications of intra-species cooperation. For instance, cooperative hunting can change how predator-prey models behave, and may resulting in intricate patterns with several periodic cycles [15-17, 23-26]. So, including the function of cooperation in ecological and eco-epidemiological systems is essential to study and analysis species interactions and population dynamics. Preys' disease and the collaborative hunting efforts of predators can result in heightened fear among prey and an increased likelihood of being captured by predators. Therefore, the presence of prey refuge plays a crucial role in the field of mathematical biology, as it can be viewed as a form of anti-predator behavior that helps safeguard against the extinction of prey populations [1,2,6,8,9,26-28]. Motivated by aforementioned literature and the results thereof, in this paper, we investigate a prey-predator model in which predator subject to hunting cooperation, while diseased prey subject to the anti-predator behaviors such as the fear impact and prey refuge. The theoretical and numerical results of this study can reveal how the interactions between disease transition, fear effect, prey refuge and hunting cooperation influence the dynamic structure of the system.

2 Mathematical model formulation

Numerous research papers have proposed models that explore the dynamics between preys and predators, specifically focusing on the presence of diseases within prey species. These studies have also investigated the impact of fear on the growth of susceptible prey and its role in reducing interactions between susceptible and infected individuals [13-17]. Authors in [14, 15, 17] assume the predator eating both healthy and ill preys, and hence they used the cost of fear on the healthy preys growth and their contact with ill preys. In [16] the authors assume the predator eating only ill preys and they used the cost of fear only on the growth of healthy preys. While in [13] the authors used the predation assumption as in [16] but the impact of fear as in [14, 15, 17]. In our study, as in [13, 16] we make the assumption that predators exclusively consume infected prey, which leads us to incorporate fear as a factor influencing only the contact between infected and susceptible individuals. Additionally, we introduce the concept of infected prey seeking refuge and exhibiting anti-predator behavior as a means of defending against predator hunting cooperation.

In this section, to construct our model, we begin by considering the principle assumptions: the populations are composed into prey species $N(T)$ and predator species $Q(T)$. Prey species also divide into susceptible prey $R(T)$ and infected prey $P(T)$, i.e $N = R + P$. Infected population $P(T)$ is unable to recover or build immunity, and the disease does not spread to predators via eating or any other methods. According to Holling-two functional response predator population $Q(T)$ exclusively consuming infected prey, where the disease infection renders the prey weak and vulnerable. Further, only infected prey has prey refuge and fear effect, while predator has hunting cooperation. Furthermore, the disease spreads occurs when infected prey comes into contact with susceptible prey, based on saturated incident rate. Therefore, from the aforementioned assumptions, the following eco-epidemiological model is derived.

$$\frac{dR}{dT} = rR\left(1 - \frac{R+P}{K}\right) - \frac{\beta RP}{(1+mP)(1+K_1Q)} - d_1R$$
$$\frac{dP}{dT} = \frac{\beta RP}{(1+mP)(1+K_1Q)} - \frac{(a+Q)(1-\theta)P}{C+(1-\theta)P} - d_2P \quad (1)$$
$$\frac{dQ}{dT} = \frac{e(a+Q)(1-\theta)P}{C+(1-\theta)P} - d_3Q$$

with initial $R(0) > 0, P(0) > 0, Q(0) > 0$. In this system, r is the susceptible prey logistic growth rate and K is the total prey environmental carrying capacity. β is the infection force and a is the predation rate. The terms $\frac{1}{1+mP}$ and $\frac{1}{1+K_1Q}$ are modeling the saturated incident rate and the fear effect, where m and K_1 represent saturation and fear factors, respectively. θ is infected prey refuge and e is conversion efficiency of Q on P. The death rates of susceptible prey, infected prey, and predator are given by d_1, d_2, and d_3, respectively. Now the next dimensionless variables are applied in the system (1):

$x_s = \frac{R}{K}$, $x_i = \frac{P}{K}$, $y = \frac{Q}{K}$ and $t = rT$.

Then (after some simplification) system (1) takes the form:

$$\frac{dx_s}{dt} = x_s(1 - x_s - x_i) - \frac{\alpha x_s x_i}{(1+Mx_s)(1+k_2y)} - D_1x_s = x_sf_1(x_s, x_i, y)$$
$$\frac{dx_i}{dt} = \frac{\alpha x_s x_i}{(1+Mx_s)(1+k_2y)} - \frac{\gamma(A+y)(1-\theta)x_iy}{\eta+(1-\theta)x_i} - D_2x_i = x_if_2(x_s, x_i, y) \quad (2)$$
$$\frac{dy}{dt} = \frac{\gamma_1(A+y)(1-\theta)x_iy}{\eta+(1-\theta)x_i} - D_3y = yf_3(x_s, x_i, y)$$

where, $k_2 = KK_1$, $\alpha = \frac{\beta K}{r}$, $M = mK, D_1 = \frac{d_1}{r}$, $\gamma = \frac{K}{r}$, $\eta = \frac{C}{K}$,

$A = \frac{a}{K}$, $D_2 = \frac{d_2}{r}$, $D_3 = \frac{d_3}{r}$, $\gamma_1 = \frac{eK}{r}$.

In next section, we shall discuss the positivity and boundedness of the system (2) to ensure that the model will behaved.

3 Positivity invariance and boundedness

Theorem 1: Any solution of system(2) starting at R_+^3 remain positive of all time.

Proof: For convenient, system (2) may be written as:

$$X' = f(X,t) = \begin{pmatrix} x_sf_1(x_s, x_i, y) \\ x_if_2(x_s, x_i, y) \\ yf_3(x_s, x_i, y) \end{pmatrix} \quad (3)$$

where, $X(t) = (x_s(t), x_i(t), y(t))^T$. Here, $f_i, i = 1, 2, 3$ are smooth functions. Hence readily can find that:

$$X(t) = X(0)e^{\left[\int_0^t f_i(x_s(\tau), x_i(\tau), y(\tau))d\tau\right]}.$$

This demonstrates that with initial condition $X(0) > 0$, we have all solution $X(t) > 0$ of (3) for $t > 0$ is positive. Hence the theorem is proved ■

Theorem 2: Each solution of the system (2) starting at R_+^3 is uniformly bounded if $\gamma_1 < \gamma$.

Proof: Let $(x_s(t), x_i(t), y(t))$ be any solution of the system (2) . Since

$\frac{dx_s}{dt} \leq x_s(1 - x_s)$,

hence can have $\lim_{t\to\infty} \sup x_s(t) \leq 1$. Now, if choosing

$$H = x_s + x_i + y, \quad (4)$$

as any solution of (2), then the derivative of Eq.(4) gives:

$$\frac{dH}{dt} = \frac{x_s}{dt} + \frac{x_i}{dt} + \frac{y}{dt}.$$

Further, if using $\psi = \min(1, D_2, D_3)$, then can have

$$\frac{dH}{dt} \leq 2x_s - \psi H,$$

or

$$\frac{dH}{dt} + \psi H \leq 2. \tag{5}$$

Hence, according to Gronwall's lemma, it follows that

$$0 < H(x_s, x_i, y) \leq H(x_s(0), x_i(0), y(0))e^{-\psi t} + \frac{2}{\psi}(1 - e^{-\psi t})$$

Consequently, using the limit as $t \to \infty$, may have $0 < H \leq \frac{2}{\psi}$.

Thus, all solutions of the system (2) enters into the region

$\Gamma = \{(x_s, x_i, y) : 0 < H \leq \frac{2}{\psi} + \epsilon$, for any $\epsilon > 0\}$ ∎

4 Extinction scenarios

This section deals with the conditions for which both the species (prey or predator) of the underlying system (2) will be going to extinct in long run. Suppose:

$\bar{y} = \lim_{t\to\infty} \sup y(t)$ and $\underline{y} = \lim_{t\to\infty} \inf y(t)$

Here we shall use following fact (for large time t): $x_s(t), x_i(t) \leq 1$.

Theorem 3: If $D_1 > 1$, then $\lim_{t\to\infty} x_s(t) = 0$.

Proof: We possess

$$\begin{aligned}\frac{dx_s}{dt} &= x_s(1 - x_s - x_i) - \frac{\alpha x_s x_i}{(1+Mx_s)(1+k_2 y)} - D_1 x_s \\ &\leq x_s(1 - x_s - x_i) - D_1 x_s \\ &\leq x_s(1 - D_1) < 0.\end{aligned}$$

Hence, $\lim_{t\to\infty} sup\ x_s(t) = 0$ if $1 < D_1$. ∎

Theorem 4: If $\alpha < D_2$, then $\lim_{t\to\infty} x_i(t) = 0$.

Proof: We possess

$$\begin{aligned}\frac{dx_i}{dt} &= \frac{\alpha x_s x_i}{(1 + Mx_s)(1 + k_2 y)} - \frac{\gamma(A + y)(1 - \theta)x_i y}{\eta + (1 - \theta)x_i} - D_2 x_i \\ &\leq \frac{\alpha x_s x_i}{(1+Mx_s)(1+k_2 y)} - D_2 x_i \\ &\leq x_i(\alpha - D_2) < 0.\end{aligned}$$ ∎

Theorem 5: If $\alpha < \min\{D_3, D_2\}$, then $\lim_{t\to\infty} y(t) = 0$.

Proof: From last two equations in system (2), we possess

$$\begin{aligned}\frac{d}{dt}(x_i + y) &= \frac{\alpha x_s x_i}{(1+Mx_s)(1+k_2 y)} - D_2 x_i - D_3 y, \text{ since } \gamma_1 < \gamma \\ &< \alpha x_s x_i - D_2 x_i - D_3 y \\ &< \alpha(x_i + y) - D(x_i + y), \text{ where } D = \min\{D_3, D_2\}, \\ &< (\alpha - D)(x_i + y)\end{aligned}$$

Therefore, $$\begin{aligned}(x_i(t) + y(t)) &= (x_i(t_0) + y(t_0))exp\left[\int_{t_0}^{t}(\alpha - D)ds\right] \\ &\leq \left(x_i(t_0) + y(t_0)\right)exp[(\alpha - D)(t + t_0)].\end{aligned}$$

Using the limit as $t \to \infty$ gives $\lim_{t\to\infty}(x_i(t) + y(t)) = 0$, provided $\alpha < D$.

Thus, $\lim_{t\to\infty} x_i(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$. ∎

5 Existence of equilibrium points

The system of Equations has the following Equilibrium Points:

1. The trivial equilibrium :$\mathcal{E}_0\ (0,0,0)$.
2. The axial equilibrium: $\mathcal{E}_1\ (\hat{x}_s,0,0)$, where $\hat{x}_s = 1 - D_1$, provided that
$$D1 < 1. \tag{6}$$
3. The predator-free boundary equilibrium: $\mathcal{E}_2\ (\bar{x}_s,\bar{x}_i,0)$, can be get from solving the following two nonlinear equations:

$$(1-\bar{x}_s-\bar{x}_i)-\frac{\alpha\bar{x}_i}{(\ 1+M\ \bar{x}_i)}-D_1=0, \tag{7}$$
$$\frac{\alpha\bar{x}_s}{(\ 1+M\ \bar{x}_i)}-D_2=0\,. \tag{8}$$

Eq.(8) gives :

$$\bar{x}_s=\frac{D_2}{\alpha}\ (\,1+M\ \bar{x}_i) \tag{9}$$

Further, with the help of Eqs.(7) and (9), it follows that

$$\zeta_1\ \bar{x}_i{}^2+\zeta_2\bar{x}_i+\zeta_3=0, \tag{10}$$

where

$$\zeta_1=M(D_2M+\alpha)\ ,\zeta_2=\alpha^2+\frac{\zeta_1}{M}+M\zeta_3,\quad \zeta_3=D_2-\alpha(1-D_1).$$

Equation (10) possess the positive solution $\bar{x}_i=\frac{-\ \zeta_2+\sqrt{\zeta_2{}^2-4\zeta_1\zeta_3}}{2\zeta_1}$, if the next condition achieved:

$$D_2 < \alpha(1-D_1). \tag{11}$$

4. Interior equilibrium: $\mathcal{E}_3\ (x_s{}^*,x_i{}^*,y^*)$ is existed uniquely in the interior of the first octant of $\mathrm{x}_s\mathrm{x}_i\mathrm{y}-$ space if and only if there is positive solution of the following set equations

$$(1-x_s-x_i)-\frac{\alpha x_i}{(1+Mx_s)(1+k_2y)}-D_1=0 \tag{12}$$
$$\frac{\alpha x_s}{(1+Mx_s)(1+k_2y)}-\frac{\gamma(A+y)(1-\theta)y}{\eta+(1-\theta)x_i}-D_2=0 \tag{13}$$
$$\frac{\gamma_1(A+y)(1-\theta)x_i}{\eta+(1-\theta)x_i}-D_3=0 \tag{14}$$

using (14), x_i can be determined by

$$x_{i1}=\frac{\mathrm{D}_3\eta}{(1-\theta)(\gamma_1\mathrm{A}+\gamma_1\mathrm{y}-\mathrm{D}_3)}=\mathfrak{h}_0\ (y) \tag{15}$$

Note that $x_{i1}>0$, when the following inequality holds:

$$D_3 < \gamma_1A. \tag{16}$$

Let, $\rho_1 = 1 + M\mathfrak{h}_0\ (y), \rho_2 = \eta + (1-\theta)\mathfrak{h}_0\ (y)$, and $\rho_3 = 1 + M\mathfrak{h}_0\ (y)$.

Then the next two isoclines are produced by substituting (15) in Eqs. (13) and (12) at respectively.

$$\mathfrak{h}_1(y,x_s) = \alpha x_s\rho_2 - \rho_1(1+k_2y)[\gamma(1-\theta)(A+y)y+D_2\rho_2] = 0 \tag{17}$$
$$\mathfrak{h}_2(y,x_s) = \rho_1(1+k_2y)(1-D_1-x_s-\mathfrak{h}_0(y))-\alpha\mathfrak{h}_0(y) = 0. \tag{18}$$

By (17), when $y \to 0$, we have $x_s \to x_{s1}$, where x_{s1} is determined by

$$x_{s1}=\frac{D_2}{\alpha}\rho_3 \tag{19}$$

Further, by (18), we notice that when $y \to 0$, we have $x_s \to x_{s2}$where the value of x_{s2} is set by:

$$x_{s2} = 1 - D_1 - \mathfrak{h}_0(0)\left(1+\frac{\alpha}{\boldsymbol{\rho_3}}\right). \tag{20}$$

If the following condition, as well as condition (16), are fulfilled, it is evident that x_{s2} is positive.

$$D_1+\ \mathfrak{h}_0(0)\left(1+\frac{\alpha}{\rho_3}\right)<1\,. \tag{21}$$

Furthermore, if the following requirements are met, then the above two isoclines have a unique intersection point in the positive quadrant of the $x_s y - plane$, indicated by $({x_s}^*, y^*)$.

$$\begin{cases} \frac{dy}{dx_s} = -(\frac{\partial \mathfrak{h}_1}{\partial x_s})/(\frac{\partial \mathfrak{h}_1}{\partial y}) > 0, \\ \frac{dy}{dx_s} = -(\frac{\partial \mathfrak{h}_2}{\partial x_s})/(\frac{\partial \mathfrak{h}_2}{\partial y}) < 0, \\ x_{s2} > x_{s1}. \end{cases} \quad (22)$$

That is, system (2) has a unique positive equilibrium point$({x_s}^*, {x_i}^*, y^*)$, where ${x_i}^*$set by:

$${x_i}^* = \frac{D_3 \eta}{(1-\theta)(\gamma_1 A + \gamma_1 y^* - D_3)} \quad (23)$$

6 Stability analysis of equilibrium points

In this section, we investigate the stability properties of the system (2) near the equilibriums $\mathcal{E}_i$; for $i = 0, 1, 2, 3$ where that are examined by constructing the Jacobian matrix relating to every equilibrium point. The Jacobian matrix of (2) at any arbitrary point (x_s, x_i, y), can be written as:

$$J(x_s, x_i, y) = (v_{ij})_{3\times 3} \quad (24)$$

where,

$v_{11} = 1 - 2x_s - x_i - \frac{\alpha x_i}{(1+Mx_s)(1+k_2 y)} - D_1$, $v_{12} = -x_s - \frac{\alpha x_i}{(1+Mx_s)^2(1+k_2 y)}$ $v_{13} = \frac{\alpha k_2 x_s x_i}{(1+Mx_s)(1+k_2 y)^2}$, $v_{21} = \frac{\alpha x_i}{(1+Mx_s)(1+k_2 y)}$, $v_{22} = \frac{\alpha x_i}{(1+Mx_s)(1+k_2 y)} - \frac{\gamma \eta (A+y)(1-\theta) y}{(\eta + (1-\theta) x_i)^2} - D_2$, $v_{23} = \frac{-\alpha k_2 x_s x_i}{(1+Mx_s)(1+k_2 y)^2} - \frac{\gamma (A+2y)(1-\theta) x_i}{\eta + (1-\theta) x_i}$

$v_{31} = 0$, $v_{32} = \frac{\gamma_1 \eta (A+y)(1-\theta) y}{(\eta + (1-\theta) x_i)^2}$, $v_{33} = \frac{\gamma_1 (A+2y)(1-\theta) x_i}{\eta + (1-\theta) x_i} - D_3$.

Theorem 6: The trivial equilibrium point ε_0 is unstable saddle point if $D_1 < 1$.

Proof: At $\mathcal{E}_0$ the variational matrix $v(\mathcal{E}_0)$ is given by

$$v(0,0,0) = \begin{pmatrix} 1 - D_1 & 0 & 0 \\ 0 & -D_2 & 0 \\ 0 & 0 & -D_3 \end{pmatrix}$$

Obviously, $v(\mathcal{E}_0)$ has three eigenvalues given by: $1 - D_1, -D_2, -D_3$. Hence, the trivial equilibrium $\mathcal{E}_0$ of system (2) is unstable saddle point. Moreover, if illogical condition $D_1 > 1$ holds, then $\mathcal{E}_0$ is an asymptotically stable point. ∎

Theorem 7: The axial equilibrium point $\mathcal{E}_1$ is locally asymptotically stably provided that:

$$\widehat{x_s} < \frac{D_2}{\alpha}. \quad (25)$$

Proof: At $\mathcal{E}_1$ the variational matrix $v(\mathcal{E}_1)$ is given by

$$v(\widehat{x_s}, 0, 0) = \begin{pmatrix} -\widehat{x_s} & -(1+\alpha)\widehat{x_s} & 0 \\ 0 & \alpha \widehat{x_s} - D_2 & 0 \\ 0 & 0 & -D_3 \end{pmatrix}.$$

The above triangular matrix has three eigenvalues given by:

$$\lambda_1 = -\widehat{x_s}, \lambda_2 = \alpha \widehat{x_s} - D_2, \lambda_3 = -D_3.$$

Eigenvalue λ_2 is negative if condition (25) holds, then ε_1 is asymptotically stable point for system (2). Otherwise $\mathcal{E}_1$ is a saddle point. ∎

Theorem 8: The predator-free boundary equilibrium point $\mathcal{E}_2$ is asymptotically stable if the next inequality meets:

$$\overline{x_i} < \frac{D_3 \eta}{(1-\theta)[\gamma_1 A - D_3]}. \quad (26)$$

Proof: At $\mathcal{E}_2$ the variational matrix $v(\mathcal{E}_2)$ is given by:

$$v(\overline{x_s},\overline{x_\iota},0)=\begin{pmatrix} -\overline{x_s} & -\left(\overline{x_s}+\dfrac{D_2}{1+Mx_i}\right) & D_2k_2\overline{x_\iota} \\ \dfrac{D_2\overline{x_\iota}}{\overline{x_s}} & -\dfrac{\alpha M x_s x_i}{(1+Mx_i)^2} & -D_2k_2\overline{x_\iota} \\ 0 & 0 & \dfrac{\gamma_1 A(1-\theta)\overline{x_\iota}}{\eta+(1-\theta)\overline{x_\iota}}-D_3 \end{pmatrix}=\left[\omega_{ij}\right]_{3\times3}.$$

The characteristic equation is computed for the above matrix as:

$$(\lambda^2-\hbar_1\lambda-\hbar_2)(\lambda-\omega_{33})=0, \tag{27}$$

where,

$$\hbar_1=\omega_{11}+\omega_{22}\,,\qquad \hbar_2=\omega_{12}\omega_{21}-\omega_{11}\omega_{22}$$

From Eq.(27), with the help of Routh Hurwitz's criterion, we may note that the roots of the quadratic polynomial $\lambda^2-\hbar_1\lambda-\hbar_2=0$ yield two eigenvalues for $v(\mathcal{E}_2)$ with negative real parts, $\lambda_{2,3}=(\hbar_1\pm\sqrt{{\hbar_1}^2+4\hbar_2})/2$. Additionally, with the aid of the condition (26), it can be shown that the third eigenvalue $\lambda_3=\omega_{33}$ has negative value. That is, $\mathcal{E}_2$ is an asymptotically stable point for the system (2) if condition (26) is satisfied, otherwise, it is a saddle point. ■

Theorem 9. Assume the conditions listed below are met, then interior equilibrium point $\mathcal{E}_3$ of the system (2) is locally asymptotically stable,

$$\vartheta_{33}<\min\{-\vartheta_{11},-\vartheta_{22}\} \tag{28}$$

$$q_3>0 \tag{29}$$

where $\vartheta_{ij}=v_{ij}$ at (x_s^*,x_i^*,y^*), while q_3 is given in the proof.

Proof. At $\mathcal{E}_3$, the variational matrix v may be reduced to:

$$v(\mathcal{E}_3)=\begin{bmatrix}\vartheta_{11} & \vartheta_{12} & \vartheta_{13}\\ \vartheta_{21} & \vartheta_{22} & \vartheta_{23}\\ 0 & \vartheta_{32} & \vartheta_{33}\end{bmatrix}. \tag{30}$$

Also, its characteristic equation may be like this:

$$\lambda^3+q_1\lambda^2+q_2\lambda+q_3=0. \tag{31}$$

Here we have

$$\begin{aligned} q_1&=-(\vartheta_{11}+\vartheta_{22}+\vartheta_{33}),\\ q_2&=\vartheta_{11}\vartheta_{22}+\vartheta_{11}\vartheta_{33}+\vartheta_{22}\vartheta_{33}-\vartheta_{32}\vartheta_{23}-\vartheta_{21}\vartheta_{12},\\ q_3&=\vartheta_{11}\vartheta_{23}\vartheta_{32}+\vartheta_{12}\vartheta_{21}\vartheta_{33}-\vartheta_{11}\vartheta_{22}\vartheta_{33}-\vartheta_{21}\vartheta_{13}\vartheta_{23}, \end{aligned}$$

where,

$$\vartheta_{11}=-x_s^*<0\quad,\quad \vartheta_{12}=-x_s^*-\frac{\alpha x_s^*}{(1+Mx_i^*)^2(1+k_2y^*)}<0,$$

$$\vartheta_{13}=\frac{\alpha k_2 x_i^* x_s^*}{(1+Mx_i^*)(1+k_2y^*)^2}>0,\qquad \vartheta_{21}=\frac{\alpha x_i^*}{(1+Mx_i^*)(1+k_2y^*)}>0$$

$$\vartheta_{22}=\frac{\alpha x_s^*}{(1+Mx_i^*)^2(1+k_2y^*)}-\frac{\eta\gamma(A+y^*)(1-\theta)y^*}{(\eta+(1-\theta)x_i^*)^2}-D_2<0$$

$$\vartheta_{23}=\frac{-\alpha x_s^* x_i^* k_2}{(1+Mx_i^*)(1+k_2y^*)^2}-\frac{\gamma(1-\theta)(A+2y^*)x_i^*}{\eta+(1-\theta)x_i^*}<0$$

$$\vartheta_{31}=0\,,\quad \vartheta_{32}=\frac{\eta\gamma_1(1-\theta)(A+y^*)y^*}{(\eta+(1-\theta)x_i^*)^2}>0$$

$$\vartheta_{33}=\frac{\gamma_1(1-\theta)x_i^* y^*}{\eta+(1-\theta)x_i^*}<0.$$

Furthermore,

$$\begin{aligned}\Delta= q_1q_2-q_3&=(\vartheta_{22}+\vartheta_{33})[\vartheta_{23}\vartheta_{32}-(\vartheta_{11}+\vartheta_{22})(\vartheta_{11}+\vartheta_{33})]\\ &\quad+\vartheta_{12}\vartheta_{21}(\vartheta_{11}+\vartheta_{22}+\vartheta_{33})+\vartheta_{21}\vartheta_{13}\vartheta_{32}.\end{aligned}$$

Now, according to the signs of $v(\mathcal{E}_3)$ elements with the help of conditions (28) and (29), one can get q_1, q_3, and all terms of Δ are positive. Hence, due to Routh-Hurwitz's criterion all roots (eigenvalues) of (31) have negative real parts. That is, $\mathcal{E}_3$ is local asymptotically stable point for sys. (2). ■

7 Numerical analysis and discussion

Some numerical simulations are offered in this part to back up our analytical and mathematical findings. These simulations reveal a better insight of the suggested model, also the system's complicated behavior. In this section, we will use a hypothetical collections of data in several examples, and numerical solutions are run using Matlab (8.1) software to back up our earlier findings.

Example 1: In the first, to discussion the existence and stability of $\mathcal{E}_3$, we consider the following hypothetical data as parametric values of model (2):

$$k_2 = 6, \ \ \alpha = 0.75, \ \ M = 0.6, D_1 = 0.05, D_2 = 0.105, D_3 = 0.05,$$
$$\gamma = 1, \ \ \gamma_1 = 0.75, \ \ A = 0.15, \ \ \eta = 0.2, \ \ \ \theta = 0.615. \quad (32)$$

With (32) data set we notice that all condition of existence ε_3 are satisfied, where $\gamma_1 A = 0.1125 > D_3$, $D_1 + \hbar_0(0)\left(1 + \frac{\alpha}{\rho^3}\right) = 0.6376 < 1$, and $x_{s1} = 0.2536 < x_{s2} = 0.3624$. Here, as plotted in Figure 1, $\hbar_1$ and $\hbar_2$ intersect at a unique positive point $(x_s^*, \ y^*) = (0.7769, 0.1639)$, and equation (3.18) gives $x_i^* = 0.1401$. Hence, we say the model (2) has the equilibrium solution $\mathcal{E}_3 = (0.7769, \ 0.1401, 0.1639)$. For checking the local stability of $\mathcal{E}_3$ by applying Theorem 9 one can find (28) and (29) conditions are fulfilled, where $v_{11} = -0.7769, v_{22} = -0.1356$, $v_{33} = 0.0261$ and $q3 = 0{:}\,0048$, i.e. $v_{33} < \min\{v_{11}, -v_{22}\}$ and $q3 > 0$. Moreover, all Routh Hurwitz's conditions also fulfilled, where $q_1 = 0.8865, q_2 = 0{,}1238, q_3 = 0{,}0048$, and $\Delta = 0.1049$. Therefore, the local asymptotically stability near the coexistence equilibrium point ε_3 is observed.

In Figure 2, the solutions time series of system (2) with different initial points near $\mathcal{E}_3$ are presented in a graphical presentation. Here, it is found that (with set of data (32)) all the solutions corresponding to these initial points converge to interior solution $(0.7769, 0.1401, 0.1639)$. Here one may noting that, a conditionally stable positive equilibrium $\mathcal{E}_3$ where all preys (susceptible x_s, infected x_i), and predator are present possess many ecological prominence. Ecologically, this equilibrium refers that there is a balanced coexistence of x_s-species, x_i-species, and y-species. The presence of predator and infected prey mean the combined effect of hunting cooperation and anti-predator behavior (fear effect and prey refuge) can preventing predator and infected prey populations to extinction. This balance shows that thereby averting the prominence of disease outbreaks, the predator can control transmission within the prey population.

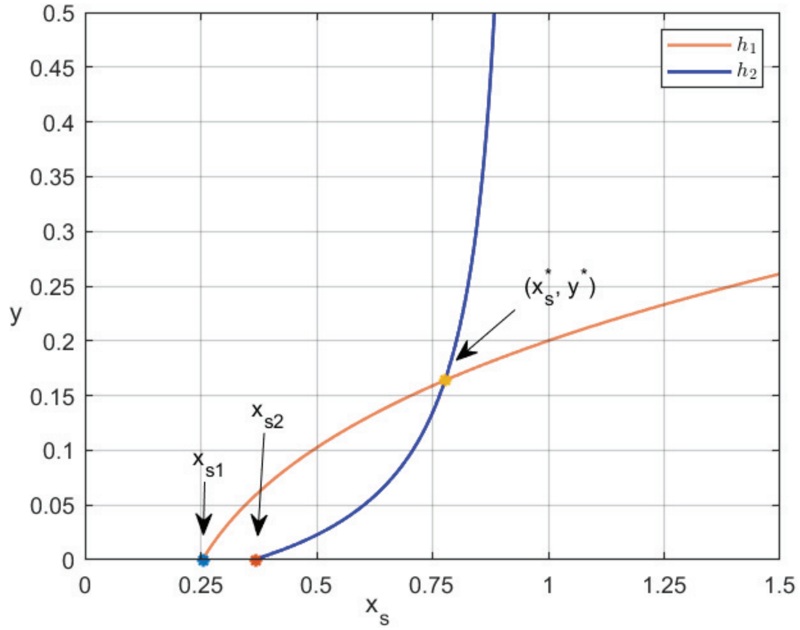

Fig. 1. Time series of system (2) for the data (32). System (2) approaches asymptotically to the uk-axis equilibrium point $\mathcal{E}_3 = (0.7769, 0.1401, 0.1639)$.

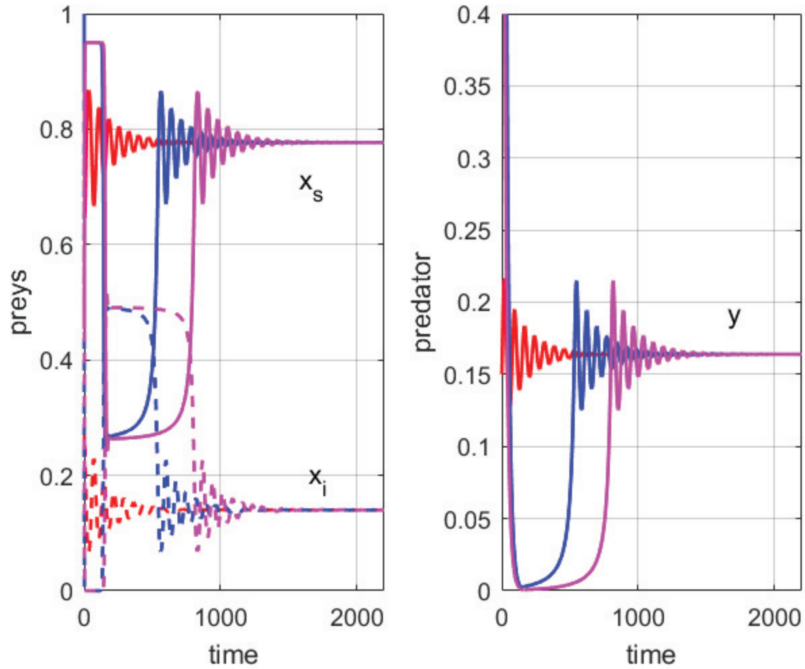

Fig. 2. Time series of system (2) for the data (32). System (2) approaches asymptotically to the interior equilibrium $\mathcal{E}_3$= (0.7769, 0.1401, 0.1639).

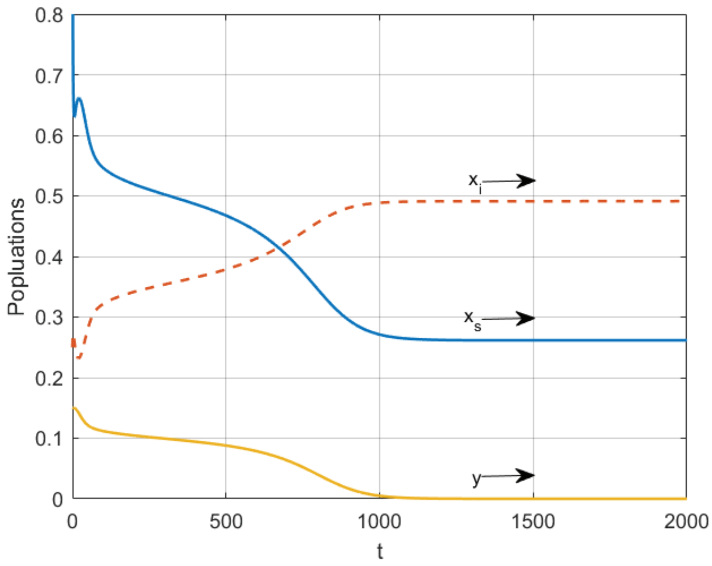

Fig. 3. Time series of system (2) for the data (32). System (2) approaches asymptotically to equilibrium solution $\mathcal{E}_2$ = (0.2616, 0.4912, 0).

Example 2 In this example, we used the set of parameters values (32) with $\gamma_1 = 0.49$ to demonstrate the existence and stability of $\mathcal{E}_2$.

Here, we can see that the condition (11) holds where $D_2 < \alpha(1 - D_1) = 0.475$, and the equation (10) take the form $0.2762\bar{x}_i^2 + 0.6175\bar{x}_i - 0.37 = 0$. Consequently, $\bar{x}_i = 0.4912$ and $\bar{x}_s$ =0.2616, and then the equilibrium solution $\mathcal{E}_2$ = (0.2616, 0.4912, 0) of model (2) is obtained. Moreover, at $\mathcal{E}_2$, the condition (26) of Theorem 3 is satisfied as $\frac{D_3\eta}{(1-\theta)[\gamma_1 A - D_3]} = 1.1053 > \bar{x}_i$, also, matrix $v(\mathcal{E}_2)$ has three eigenvalues with negative real parts given by $\lambda_1 = -0.0143$, $\lambda_2 = -0.1411 + 0.22317i$, $\lambda_3 = -0.1411 - 0.2317i$, and consequently $\mathcal{E}_2$ is asymptotically stable as plotted in Figure 3.

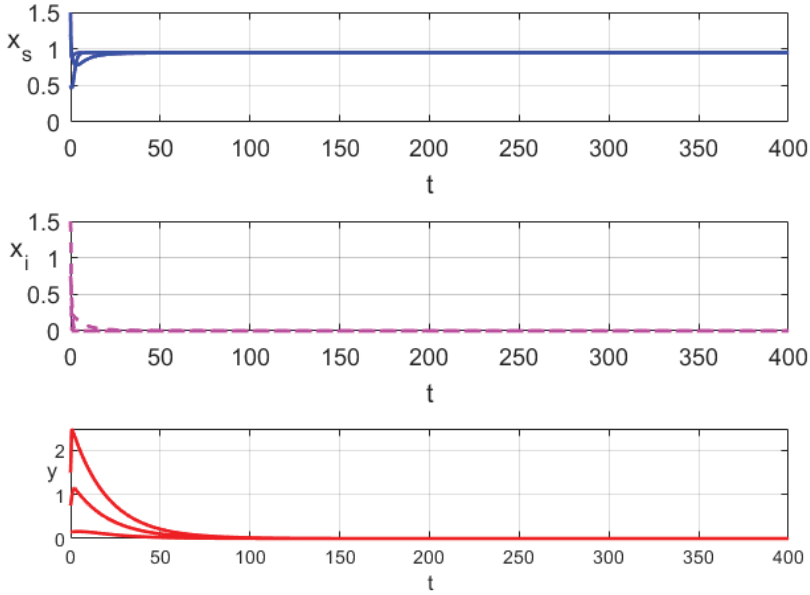

Fig. 4. Time series of system (2) for the data (32) with α= 0.11. System (2) approaches asymptotically to equilibrium solution $\mathcal{E}_1 = (0.95, 0, 0)$.

Example 3: In this example, for confirming the existence and stability of susceptible prey equilibrium ε_1, we used α= 0.11 and the other parameter values as given in the data set (32).

As can be seen from the data set utilized in this example, the requirement of existence $\mathcal{E}_1$ is valid, and model (2) has the unique equilibrium solution $\mathcal{E}_1 =$ (0.95, 0, 0). Additionally, we have $\frac{D_2}{\alpha} = 0.9545 > x_s = 095$, i.e. the local stability requirement (25) in Theorem 7 is also valid. Moreover, matrix $v(\mathcal{E}_2)$ has only negative eigenvalues given by $\lambda_1 = -0.9500, \lambda_2 = -0{:}\,0005, \lambda_3 = -0{:}\,0500$. Hence, as shown in Figure.4, $\mathcal{E}_1 = (0.95, 0, 0)$ which appears to be locally asymptotically stable.

Example 4: The focus of this example is on the existence of bistability in system (2). We used $k_2 = 3.25$, $M = 0.325$ and $D3 = 0{:}\,0825$, while the remaining parameters had values similar to that of example 1.

For the set of values used in this example, we discovered that the system has two stable equilibrium points $\mathcal{E}_3$ and $\mathcal{E}_2$ simultaneously, and that, depending on the initial conditions, the system can settle at either of these points. Figure 5 show that if (1.20.25 0.15) is chosen as the initial condition, the system's solution will converge to interior equilibrium $\mathcal{E}_3$ and if the solution begins at (0.1 0.1 0.1), the system solution moves to boundary equilibrium point $\mathcal{E}_2$. This means that the system (2) can changeover their stability between these two states depending on the initial conditions. The results indicate that, for lower values of fear effect and saturated rate, coexistence depends on the initial population sizes due to the bistability between the coexistence equilibria and equilibrium of disease-free and predator-free.

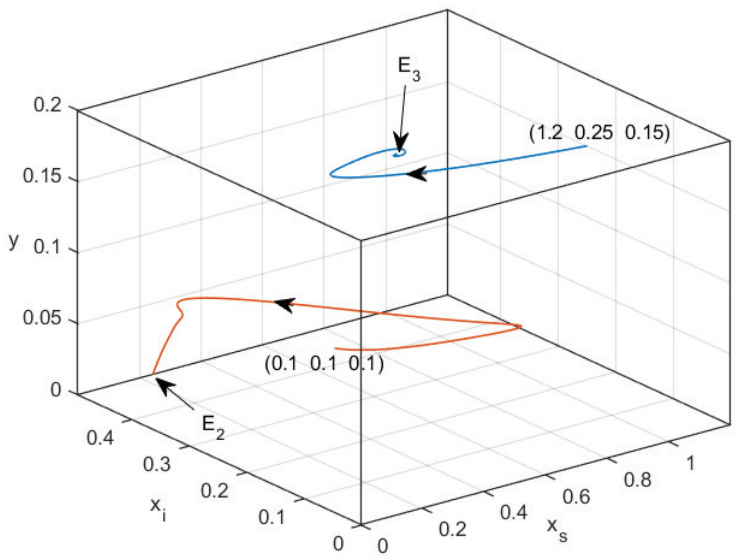

Fig. 5. Phase portrait illustrate the bistability in system (2) for data set (32) with $k_2 = 3.25$, $M = 0.325$, and $D_3 = 0.0825$.

Example 5. The last example focus on the effect of the parameters M, θ, and k_2 together on the dynamic of system (2). Here, certain values for these parameters could be used as $M = 0.12, 0.86$; $\theta = 0.3, 0.7$, and $k_2 = 3,\ 30$, while the values of the remaining parameters were like those in example 1. Figure 6 plots the numerical solutions xs, xi and y for the sets of values in this example. The top subfigures were plotted at $M = 0.12$, while the bottom subfigures were plotted at $M = 0.86$. For small values of M, θ, and k_2, the solutions in the first upper subfigures lead to a periodic solution; as a result, $\mathcal{E}_3$ loses stability, as seen in Figure.6(a). Also as shown in Figures.6(c-d), after small oscillation behavior the solutions switch to stable $\mathcal{E}_3$ for raising the value of θ and k_2. Furthermore, Figure 6(e) demonstrates that the solutions asymptotically converge to $\mathcal{E}_3$ consistently for large values of θ and k_2. Whereas, the system exhibits the same dynamic behavior (with smaller periodic size) in lower subfigures at $M = 0.86$ as seen in Figures. 6(e-g). Nevertheless, in Figure.6(h) the predator number goes to extinction as a result of increasing the values of M, θ ,and k_2 together.

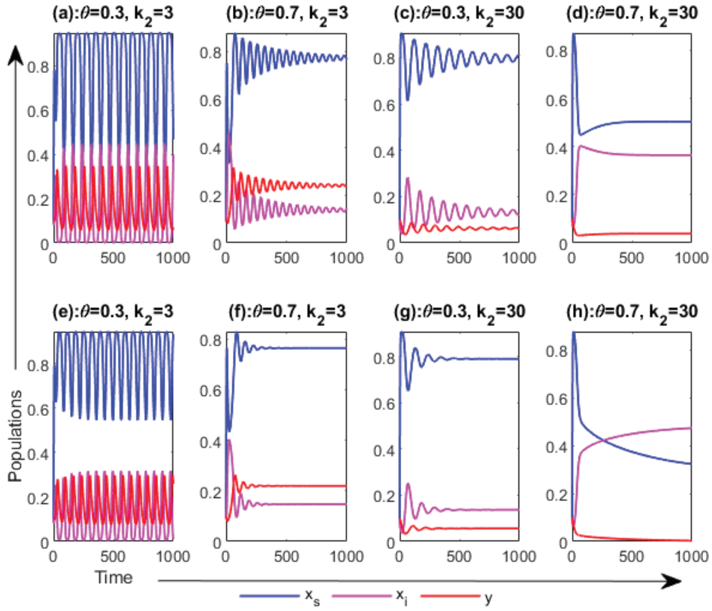

Fig. 6. Time series of system (2) for different values of M, θ and k_2.

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