On subgroups product graph of finite groups

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Abstract. This paper explores Subgroup Product Graphs (SPG) in cyclic groups, presenting a Vertex Degrees Formula based on the prime factorization of a positive integer n. The Isolated Vertex Property asserts that for a positive integer n, the SPG γ_sp(G) lacks isolated vertices. The Matrix Degree and Edge Formula provide a matrix representation and calculate the edges in SPG. Additionally, a Subgraph Relation identifies the complete graph $K_{\pi(n)}$ as a subgraph in γ_sp(G). Specific Examples illustrate vertex degrees for different n values. In essence, the study contributes isomorphisms, characterizes properties, and computes degrees and edges for diverse subgroups in Subgroup Product Graphs.

1 Introduction

The idea of associating graphs with groups originated in Kelly graphs and has developed into an important research focus in modern algebraic graph theory A notable development in recent years has been to identify and study a graph type defined in categories, such as power graphs [1,7], augmented power maps [2,3], navigation diagrams [9,10], non-navigation diagrams [1, 4 ,5], subgroup inclusion statistics [8], and so on. Many works such as [6] have been done on these topics. For a comprehensive description of group-defined records, [10] is a good reference. These graphs provide valuable insight into understanding group diversity through graph-theoretic explanation.

In this paper, we present a novel graph defined on groups and investigate its various properties and characterizations. Additionally, we introduce relevant concepts, including the notation *π(n)* for the set of all prime factors of a positive integer n.

The subgroup H of a group G is defined such that for every $a,\ b \in H$, the condition $ab^{-1} \in H$ holds. If G is a finite group, we denote by *π(G)* the proper subgroup of a set G as a maximal subgroup H if G does not have a proper subgroup in which H is correct.

In this paper we compute the generalized of product subgroup graph of group for cyclic group of order n, it was presented by Angsuman Das with authors in 2021, as well as, we compute some of properties of line graph, complete graph of this graph, so we presenting the commuting probability for subgroups of a cyclic group.

Consider a finite institution G. A subgroup H of G earns the designation "self-normalizing in G" whilst its normalizer inside G, denoted as $N_G(H)$, is same to H itself. It is clear that a maximal subgroup of G is both a normal subgroup or self-normalizing. The notations

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Max(G), NMax(G), and SNMax(G) constitute the sets of all maximal subgroups, regular maximal subgroups, and self-normalizing maximal subgroups of G, respectively. Additionally, we rent |Max(G)|, |NMax(G)|, and |SNMax(G)| to denote the counts of those various subgroup kinds.

If H is a subset of the group G, the notation $< H_i >$ denotes the subgroup generated by H. Within G, L(G) is the intersection of self-normalizing maximal subgroups, denoted as $N_G(H) = H$, while R(G) represents the intersection of normal maximal subgroups of G. In both instances, L(G) or R(G) is considered as G if the corresponding maximal subgroups do not exist properly. The Frattini subgroup, denoted by (G), is isomorphic to the intersection of L(G) and R(G). The set of prime factors for a positive integer n is denoted by π(n). Shelash and Ashrafi presented the count of maximal subgroups for certain finite groups in 2019.

The number of all maximal subgroups of the cyclic group be equal to $Max(C_n) = \pi(n)$ and they are $< a^{\pi} >$ for each $i \in \pi(n)$. Take C_{30}, it is easy to see that the $< a^2 >, < a^3 >$ $, < a^5 >$ are maximal subgroups of the group C_{30}.

Let $\gamma(V, E)$ be a finite graph, $V(\gamma)$ is a set of all vertices and $E(\gamma)$ is a set of al edges of the graph γ.

The product subgroup graph of group, is denoted by $\gamma_{sp}(G)$ (V, E) is a simple graph and its has not loop and multiplicity of edge, the set of vertices is a set of all proper of the subgroups, and for each two vertices $< H >, < K >$ is join if $< HK > \cong G$. the any vertex hs no edge is called isolated vertex.

2 Main results

In this section, we explore the characteristics of the Subgroup Product Graph derived from the cyclic group of order n. It is clear that for each element in cyclic group, its generate a subgroup H when $< a^i >$ for all $i \mid n$.

Proposition 2.1. Let C_n be a cyclic group and $H =< a^i >, K =< a^j >$ is a subgroup of C_n, $HK = G$ if $i \neq j$ and $gcd(i, j) = 1$.

3 Cyclic group C_n

In this section, we introduce some of important results about subgroup product graph of the cyclic group. Its well known the cyclic group of order is define by $C_n =< a: a^n = e >$ and the number of subgroups is given by $\tau(n)$ and they are representation by $< a^i >$ for each $i \mid n$.

Remark 2.2.

In the following, we will introduce some of the remarks.

1. The set of all subgroup is define by $Sub(G) = \{H, \forall H \leq G\}$;
2. The set of all trivial subgroups is define by $TSub(G) = \{e, G\}$;
3. The set of all proper subgroups is define by $PSub(G) = Sub(G) \setminus TSub(G)$;
4. The set of all isolated vertices is define by $IsV(\gamma_{sb}(G)) = \{v \in V(\gamma_{sb}(G)), \deg(v) = 0\}$

4 Computing algorithm degree of vertices

In this section, we outline an algorithm for computing the number degree of the vertices of the graph $\gamma_{sp}(G)$ for some of the finite group.

Proposition 2.3.

Suppose that $n=\prod_{1\le i\le r} p_i$ is a positive integer, the degree of vertices is given by the following

$$\deg(v)=\begin{cases}\deg(<a^{p}>)=2^{(r-1)}-1\\ \deg(<a^{p_ip_j}>)=2^{(r-2)}-1\\ \deg(<a^{p_ip_jp_k}>)=2^{(r-3)}-1\\ \deg(<a^{p_ip_jp_kp_h}>)=2^{(r-4)}-1\\ \vdots\\ \deg\left(<a^{\prod_{1\le i\le (r-2)} p_i}>\right)=3\\ \deg\left(<a^{\prod_{1\le i\le (r-1)} p_i}>\right)=1\end{cases}$$

Proof. Consider a finite group G, by Proposition in when H,K be two non-trivial proper subgroup of G such that $[G:H]$ and $[G:K]$ are coprime, then $HK=G$, i.e., $H\sim K\in E(\gamma_{sp}(G))$. Its clear that the subgroup of type $<a_j^p>$ have connect with all vertices of types $<a^{\prod_{1\le i\le r} p_i}>, \forall r, i\ne j$, because $\left[\prod_{1\le i\le r} p_i : \frac{\prod_{1\le i\le r} p_i}{\mathrm{pj}}\right]=p_j$ is comprime with all p_i for all i and $i\ne j$. By $\tau(n)$ function, We have the capability to calculate all subgroups of the cyclic group, determining the total number of vertices connected by edges in the process with $<a_j^p>$ be

$$\tau\left(\prod_{1\le i\le r-1} p_i\right)-1=\tau(\mathrm{p}_1)\tau(\mathrm{p}_2)\ldots\tau(\mathrm{p}_{\mathrm{r}-1})-1$$
$$=(2)(2)\ldots(2)-1$$
$$=2^{(r-1)}-1$$

By similar for remains of vertices.

Corollary 2.4.

Let G be a cyclic group of order n and $n=\prod_{1\le i\le r} p_i$ is a positive integer number, the set of $IsV(\gamma_{sb}(G))=\phi$

Proof. Direct from Proposition 2.3.

Theorem 2.5.

let $G\cong C_n$ be a cyclic group and $n=\prod_{1\le i\le r} p_i$ is a positive integer number, the matrix degree is given by the following:

$$DM_{sp}(G)=\begin{pmatrix} C_1^r & C_2^r & \cdots & C_{r-2}^r & C_{r-1}^r\\ 2^{(r-1)}-1 & 2^{(r-2)}-1 & \cdots & 3 & 1\end{pmatrix}$$

Corollary 2.6.

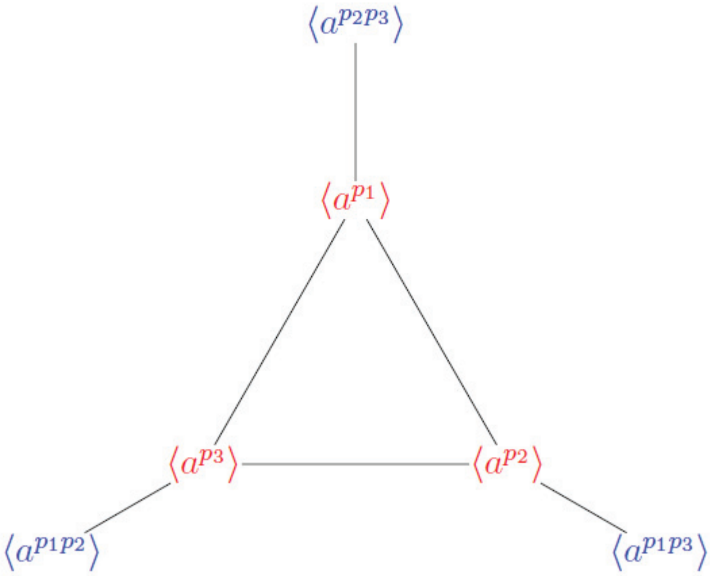
Let $G\cong C_n$ be a cyclic group and $n=\prod_{1\le i\le r} p_i$ is a positive integer number, the number of edges is given by the following:

$$E\left(\gamma_{sp}(G)\right)=\frac{\sum_{i=1}^{r-1}(2^{(r-i)}-1)\ C_i^r}{2}$$

Example 2.7.

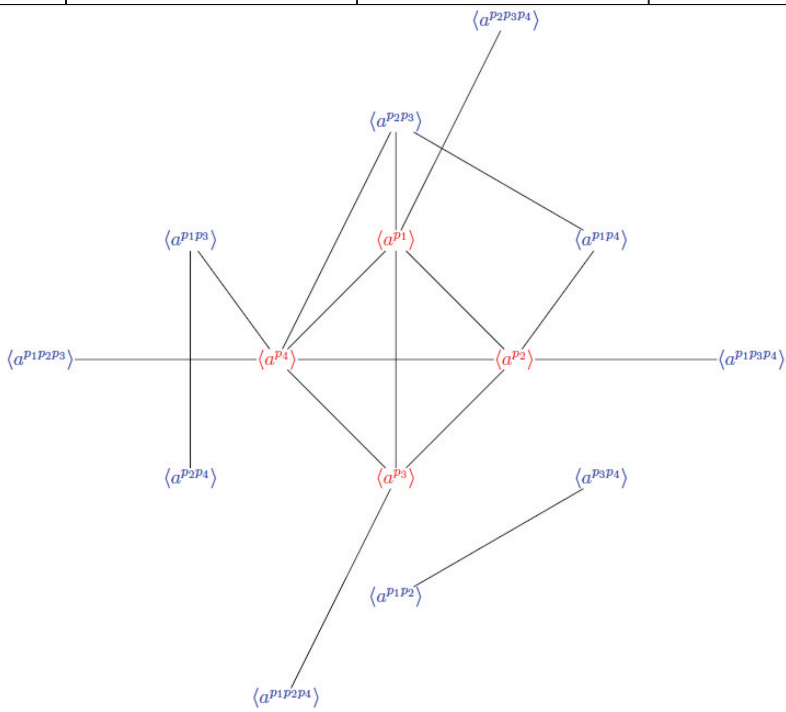
Let $G\cong C_{p_1p_2p_3}$ be a cyclic group.

v	$\deg(v)$	v	$\deg(v)$
$<a^{p_1}>$	3	$<a^{p_1p_2}>$	1
$<a^{p_2}>$	3	$<a^{p_1p_3}>$	1
$<a^{p_3}>$	3	$<a^{p_2p_3}>$	1



Example 2.8.
Let $G \cong C_{p_1p_2p_3p_4}$

v	$\deg(v)$	v	$\deg(v)$
$< a^{p_1} >$	7	$< a^{p_1p_2} >$	3
$< a^{p_2} >$	7	$< a^{p_1p_3} >$	3
$< a^{p_3} >$	7	$< a^{p_2p_3} >$	3
$< a^{p_3} >$	7	$< a^{p_1p_2} >$	3
$< a^{p_1p_2} >$	3	$< a^{p_1p_2} >$	3
$< a^{p_1p_2} >$	1	$< a^{p_1p_2} >$	1
$< a^{p_1p_2} >$	1	$< a^{p_1p_2} >$	1



Corollary 2.9.

Let $G \cong C_n$ be a cyclic group and $n = \prod_{1 \le i \le r} p_i$ is a positive integer number, the complete graph $K_{\pi(n)}$ is a subgraph in $\gamma_{sp}(G)$.

Corollary 2.10.

Let $G \cong C_n$ be a cyclic group and $n = p_i p_j$ is a positive integer number, then $\gamma_{sp}(G) = K_2$.

Example 2.11.

Take $G \cong C_{15}$. Then the $SP(C_{15})$ be isomorphic to the following graph.

$$\langle a^3 \rangle \text{ ——————— } \langle a^5 \rangle$$

5 Conclusion

In summary, this research focused on exploring Subgroup Product Graphs (SPG) within the context of cyclic groups. The findings can be encapsulated as follows:

1. Vertex Degrees Formula:

Consider a positive integer $n = \prod_{1 \le i \le r} p_i$, where p_i are prime factors. The degrees of vertices in the Subgroup Product Graph (SPG) are determined by the formula:

$$\deg(v) = \begin{cases} \deg(< a^{p} >) = 2^{(r-1)} - 1 \\ \deg(< a^{p_i p_j} >) = 2^{(r-2)} - 1 \\ \deg(< a^{p_i p_j p_k} >) = 2^{(r-3)} - 1 \\ \deg(< a^{p_i p_j p_k p_h} >) = 2^{(r-4)} - 1 \\ \quad . \\ \quad . \\ \quad . \\ \deg\left(< a^{\prod_{1 \le i \le (r-2)} p_i} >\right) = 3 \\ \deg\left(< a^{\prod_{1 \le i \le (r-1)} p_i} >\right) = 1 \end{cases}$$

2. Isolated Vertex Property:

When $n = \prod_{1 \le i \le r} p_i$ is a positive integer, the SPG $\gamma_{sp}(G)$ has no isolated vertices.

3. Matrix Degree and Edge Formula:

The matrix degree (G) is given by:

$$DM_{sp}(G) = \begin{pmatrix} C_1^r & C_2^r & \cdots & C_{r-2}^r & C_{r-1}^r \\ 2^{(r-1)} - 1 & 2^{(r-2)} - 1 & \cdots & 3 & 1 \end{pmatrix}$$

4. The number of edges $E(\gamma_{sp}(G))$ is calculated as:

$$E\left(\gamma_{sp}(G)\right) = \frac{\sum_{i=1}^{r-1} (2^{(r-i)} - 1) \; C_i^r}{2}$$

5. Subgraph Relation:

The complete graph $K_{\pi(n)}$ is a subgraph in $\gamma_{sp}(G)$.

6. Specific Examples:

For $n = p_1 p_2 p_3$, the SPG exhibits specific vertex degrees.

For $n = p_1 p_2 p_3 p_4$, a detailed representation of vertex degrees is provided.

In essence, the study establishes isomorphisms, characterizes properties, and computes degrees and edges for different subgroups within the realm of Subgroup Product Graphs.

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