BIO Web of Conferences **97**, 00158 (2024) https://doi.org/10.1051/bioconf/20249700158
ISCKU 2024

Strongly nearly-2-absorbing submodules

Shwkaea M. Rajab[1*] and *Haibat K.* Mohammadali[1]

[1]Department of Mathematics , College of Computer Science and Mathematics , Tikrit University, Tikrit, Iraq.

Abstract. Let Ɍ be a commutative ring with identity and V be a unitary left Ɍ-module. The concept of Strongly Nearly-2-Absorabing sub-modules as a generalization of Endo-2-Absorabing sub-modules and strong form of Nearly-2-Absorabing sub-modules are introduced in this paper. Many examples, basic properties of this concept are introduced. Furthermore we prove that in class of (scalar, cyclic and finitely-generated) modules the two concepts Nearly-2-Absorabing sub-modules and Strongly Nearly-2-Absorabing sub-modules are equivalent. Moreover we prove that Endo-2-Absorabing and Strongly Nearly-2-Absorabing sub-modules are equivalent in class of (semi simple, regular) modules. Also those concepts are equivalent in class of all modules over a v-ring. Finally we prove several characterizations of Strongly Nearly-2-Absorabing sub-modules in some types of modules such as (projective, faithful and content) modules in class of cyclic modules.

1 Introduction

Throw out this paper all rings are commutative with identity, and all modules are unitary left Ɍ-modules. Endo-2-Absorabing sub-modules are the famous concept to start with, and were first introduced by Harfash in (2015) as a strong of 2-Absorabing sub-modules. In this paper we introduce new generalizations of Endo-2-Absorabing sub-modules which we called Strongly Nearly-2-Absorabing sub-modules. These papers consist of three parts. In part one introduce several well-known definitions and propositions that we needed throw this paper. In part two we give some basic proposition, examples of Strongly Nearly-2-Absorabing sub-modules, and prove that every Endo-2-Absorabing is Strongly Nearly-2-Absorabing sub-modules, but not conversely. Also prove that every Strongly Nearly-2-Absorabing is Nearly-2-Absorabing sub-modules, but not conversely. Many basic properties and propositions of Strongly Nearly-2-Absorabing sub-modules are introduce in this part. In part three dell with introduced several characterizations of Strongly Nearly-2-Absorabing sub-modules in some kinds of modules.

---

*Corresponding author : ali.shebl@st.tu.edu.iq

2 Basic Concept

2.1 Definition (2.1)[1]

A proper sub-module S of an R-module V is called Endo-2-Absorabing sub-module if whenever $(fog)(v) \in S$, for $f, g \in End(V), v \in V$, then either $f(v) \in S$ or $g(v) \in S$ or $(fog)(V) \subseteq S$.

2.2 Definition (2.2)[2]

The Jacobson radical $\mathcal{J}(V)$ of an R-module V is defined as intersection of all maximal sub-modules of$V$.

2.3 Definition (2.3)[3]

"A proper sub-module S of an R-module V is called Naely-2-Absorabing, if whenever $rcv \in S$, for $r, c \in$ R, $v \in V$, implies that either $rv \in S + \mathcal{J}(V)$ or $cv \in S + \mathcal{J}(V)$ or $rcV \subseteq S + \mathcal{J}(V)$. And an ideal I of a ring R is called Naely-2-Absorabing ideal of R if I is an Naely-2-Absorabing sub-module of an R-module R".

2.4 Definition (2.4)[4]

A proper sub-module S of an R-module V is called 2-Absorabing, if whenever $rcv \in S$, for $r, c \in$ R, $v \in V$, implies that either $rv \in S$ or $cv \in S$ or $rcV \subseteq S$.

2.5 Definition (2.5)[5]

An R-module V is called semi simple if every sub-module of V is a direct summand of$V$. Equivalently V is semi simple if and only if $\mathcal{J}(V) = 0$.

2.6 Proposition (2.6)[2, Exercise (12)]

If S is a sub-module of an R-module$V$, with S is a direct summand of V, then $\mathcal{J}\left(\frac{V}{S}\right) = \frac{\mathcal{J}(V)+S}{S}$.

2.7 Definition (2.7)[6]

An R-module V is called regular if $\frac{\mathrm{R}}{ann(v)}$ is regular$\forall v \in V$.

2.8 Proposition (2.8)[6, proposition (3.9)]

Let V be a regular R.-module then $\mathcal{J}(V) = 0$.

2.9 Definition (2.9)[7]

A ring R is v-ring if for any R-module V, then $\mathcal{J}(V) = 0$.

2.10 Proposition (2.10)[2, proposition (9.1.4)(b)]

If S is a sub-module of an R-module V, with $\mathcal{J}\left(\frac{V}{S}\right) = 0$, then $\mathcal{J}(V) \subseteq S$.

2.11 Definition (2.11)[8]

An R $-$ module V is said to be scalar module if for each $f \in End(V)$, there exists $r \in$ R such that $f(v) = rv$ for each $v \in V$.

2.12 Proposition (2.12)[8, proposition (1.1.7)]

Every cyclic R $-$module V is scalar module.

2.13 Proposition (2.13)[8, corollary (1.1.11)]

If V is a finitely $-$ generated R-module, then V is scalar module.

2.14 Definition (2.14)[2]

An R-epimorphism $f: V \longrightarrow V'$ is called small epimorphism if $\ker(f)$ is small sub-module of$V$.

2.15 Definition (2.15)[9]

An R-module V is fully stable if each sub-module of V is stable.

2.16 Remark (2.16)[9, p.7]

Every stable sub-module is fully invariant.

2.17 Definition (2.17)[2]

An R-module V is called A-projective if for each homomorphism $f: V \longrightarrow W$, where W is R-module and each R-epimorphism $g: A \longrightarrow W$ there exists an R-homomrphism $h: V \longrightarrow A$ such that $goh = f$.

2.18 Definition (2.18)[2]

An R-module V is called M-injective if for each R-homomorphism $g: K \longrightarrow L$, and each R-homorphism $f: K \longrightarrow M$, where M is R-module there exists an R-homomrphism $h: M \longrightarrow V$ such that $hof = g$.

2.19 Definition (2.19)[10]

"An R-module V is multiplication if every sub-module S of V is of the form $S = IV$ for some ideal I of R. Equivalently V is multiplication if $S = [S:_{\mathrm{R}} V]V$".

2.20 Definition (2.20)[11]

"For any sub-module S and L of a multiplication R-module V with $S = IV$, $L = \mathcal{J}V$, for some ideals I and $\mathcal{J}$ of R. The product $SL = IS.\mathcal{J}L = I\mathcal{J}V$, that is $SL = IL$, in particular $SV = IVV = IV = S$".

2.21 Definition (2.21)[10]

If V is cyclic R-module, then V is a multiplication.

2.22 Proposition (2.22)[2, Theorem (9.2.1)(9)]

For any projective R-module V, then $\mathcal{J}(V) = \mathcal{J}(\mathrm{R})V$.

2.23 Definition (2.23)[2]

An R-module V is faithful if $ann(V) = \{r \in \mathrm{R}: rv = (0)\} = (0)$.

2.24 Proposition (2.24)[12, Remark p.14]

Let V be faithful multiplication R-module, then $\mathcal{J}(V) = \mathcal{J}(\mathrm{R})V$.

2.25 Definition (2.25)[13]

"An R-module V is said content if $(\bigcap_{i\in I} A_i)V = \bigcap_{i\in I} A_i V$, for some family of ideals A_i in R".

2.26 Proposition (2.26)[12, proposition (1.11)]

If V is content module then $\mathcal{J}(V) = \mathcal{J}(\mathrm{R})V$.

2.27 Proposition (2.27)[14, corollary of Theorem(9)]

"Let V be a finitely-generated multiplication R-module, I_1 and I_2 are ideals in R. Then $I_1V \subseteq I_2V$ if and only if $I_1 \subseteq I_2 + ann_{\mathrm{R}}(V)$".

3 Strongly nearly-2-absorabing sub-modules

In this part of the paper we introduce the definition of Strongly Nearly-2-Absorabing sub-module and give some basic properties, examples of this concept.

3.1 Definition (3.1)

A proper sub-module S of an R-module V is called Strongly Nearly-2-Absorabing (for short STN-2-Absorabing) sub-module of V, if whenever $(f \circ g)(v) \in S$, for $f, g \in End(V)$ and $v \in V$, implies that either $f(v) \in S + \mathcal{J}(V)$ or $g(v) \in S + \mathcal{J}(V)$ or $(f \circ g)(V) \subseteq S + \mathcal{J}(V)$.

3.2 Proposition (3.2)

Every Endo-2-Absorabing sub-module S of an Ɍ-module V is STN-2-Absorabing sub-module.

3.2.1 Proof

Let S be Endo-2-Absorabing sub-module of an Ɍ-module V and let $(f \circ g)(v) \in S$, for $f, g \in End(V)$ and $v \in V$, then $f(v) \in S \subseteq S + \mathcal{J}(V)$ or $g(v) \in S \subseteq S + \mathcal{J}(V)$ or $f(v) \in S \subseteq S + \mathcal{J}(V)$ $(f \circ g)(V) \in \ S \subseteq S + \mathcal{J}(V)$. Thus S is STN-2-Absorabing sub-module of V.

The convers of proposition(3.2) is not true in general, the following example explain that.

3.3 Example (3.3)

Consider the Z −module Z_8 and sub-module $S = \langle \bar{0} \rangle$, S is STN-2-Absorabing sub-module, because $\mathcal{J}(Z_8) = \langle \bar{2} \rangle$ but S is not Endo-2-Absorabing. Since $f, g: Z_8 \to Z_8$ defined by $f(v) = 2v$, $g(v) = 2v$ for all $v \in Z_8$, so that $(f \circ g)(2) \in \langle \bar{0} \rangle$. But $f(2) \notin \langle \bar{0} \rangle$ and $g(2) \notin \langle \bar{0} \rangle$ and $(f \circ g)(Z_8) = \{\bar{0}, \bar{4}\} \nsubseteq \langle \bar{0} \rangle$.

The convers of proposition(3.2) satisfied in the following propositions under certain conditions.

3.4 Proposition (3.4)

Let V be a semi simple Ɍ-module and S be a proper sub-module of V. Then S is Endo-2-Absorabing sub-module of V if and only if S is STN-2-Absorabing sub-module of V.

3.4.1 Proof

⇒) Directly through the proposition(3.2).

⇐) Assume $(f \circ g)(v) \in S$, where $v \in V$ and $f, g \in End(V)$. Since V is STN-2-Absorabing sub-module of V, then either $f(v) \in S + \mathcal{J}(V)$ or $g(v) \in S + \mathcal{J}(V)$ or $(fog)(V) \subseteq S + \mathcal{J}(V)$. Since V is a semi simple, then $\mathcal{J}(V) = 0$. Therefore $f(v) \in S$ or $g(v) \in S$ or $(fog)(V) \subseteq S$. Hence S is Endo-2-Absorabing sub-module of V.

3.5 Proposition (3.5)

Let V be a regular Ɍ-module and S be a proper sub-module of$V$. Then S is Endo-2-Absorabing sub-module of V if and only if S is STN-2-Absorabing sub-module of V.

3.5.1 Proof

⇒) Directly through the proposition(3.2).

⇐) Assume $(f \circ g)(v) \in S$, where $v \in V$ and $f, g \in End(V)$. Since V is STN-2-Absorabing sub-module of V, then either $f(v) \in S + \mathcal{J}(V)$ or $g(v) \in S + \mathcal{J}(V)$ or $(fog)(V) \subseteq S + \mathcal{J}(V)$. But V is regular, then by proposition(2.8) $\mathcal{J}(V) = 0$. Hence S is Endo-2-Absorabing sub-module of V.

3.6 Proposition (3.6)

Let V be an R-module over a v-ring R, and S be a proper sub-module of$V$. Then S is Endo-2-Absorabing sub-module of V if and only if S is STN-2-Absorabing sub-module of V.

3.6.1 Proof

⇒) Directly through the proposition(3.2).
⇐) Since V is an R-module over v-ring R, then $\mathcal{J}(V) = 0$. That is the proof is direct.

3.7 Proposition (3.7)

Let V be an R-module, and S be a proper sub-module of V with $\mathcal{J}(V) \subseteq S$. Then S is Endo-2-Absorabing sub-module of V if and only if S is STN-2-Absorabing sub-module of V.

3.7.1 Proof

⇒) Directly through the proposition (3.2).
⇐) Let $(f \circ g)(v) \in S$, for $v \in V$ and $f, g \in End(V)$. Since V is STN-2-Absorabing sub-module of V, then either $f(v) \in S + \mathcal{J}(V)$ or $g(v) \in S + \mathcal{J}(V)$ or $(fog)(V) \subseteq S + \mathcal{J}(V)$. Since $\mathcal{J}(V) \subseteq S$, then $\mathcal{J}(V) + S = S$. Hence the proof is followed.

The following corollary as a direct application of proposition(3.7).

3.8 Corollary (3.8)

Let V be an R-module, and S be a proper sub-module of V with $\mathcal{J}\left(\frac{V}{S}\right) = 0$. Then S is Endo-2-Absorabing sub-module of V if and only if S is STN-2-Absorabing sub-module of V.

The following propositions explain the relationships of STN-2-Absorabing sub-modules with Nearly-2-Absorabing sub-modules.

3.9 Proposition (3.9)

Every STN-2-Absorabing sub-module S of an R-module V is Nearly-2-Absorabing sub-module.

3.9.1 Proof

Let S be STN-2-Absorabing sub-module of an R-module V and let $rcv \in S$, for $r, c \in$ R and $v \in V$. Let $f, g \in End(V)$ defined by $f(v) = rv$, $g(v) = cv$. Thus $(f \circ g)(v) \in S$. Since S is STN-2-Absorabing sub-module of V, then either $f(v) \in S + \mathcal{J}(V)$ or $g(v) \in S + \mathcal{J}(V)$ or $(f \circ g)(V) \subseteq S + \mathcal{J}(V)$. Thus either $rv \in S + \mathcal{J}(V)$ or $cv \in S + \mathcal{J}(V)$ or $rc(V) \subseteq S + \mathcal{J}(V)$. S is Nearly-2-Absorabing sub-module of V.

The convers of proposition(3.9) is not true in general as the following example explain that.

3.10 Example (3.10)

Let $V = Z \oplus Z$, $\mathrm{R} = Z$ and the sub-module $S = 0 \oplus 7Z$, it is clear that S is Nearly-2-Absorabing sub-module of V. But S is not STN-2-Absorabing sub-module to show that.

Let $f\colon V \to V$ define by $f(v,n) = (n,v)$, $g\colon V \to V$ define by $g(v,n) = (n,0)$, for $v,n \in V$, consider $(f \circ g)(1,7) = f\big(g(1,7)\big) = f(7,0) = (0,7) \in S$, but $f(1,7) = (1,7) \notin S + \mathcal{J}(V)$ and $g(1,7) = (0,7) \notin S + \mathcal{J}(V)$ and $(f \circ g)(Z \oplus Z) = f\big(g(Z \oplus Z)\big) = f(Z,0) = (0,Z) \nsubseteq S + \mathcal{J}(V)$.

The convers of proposition(3.9) is hold in the following results under certain conditions.

3.11 Proposition (3.11)

Let V be a scalar Ȑ-module and S be a proper sub-module of V. Then S is a STN-2-Absorabing sub-module of V if and only if S is a Nearly-2-Absorabing sub-module of V.

3.11.1 Proof

⇒) Directly through the proposition(3.9).

⇐) Assume that $(f \circ g)(v) \in Sc$, where $v \in V$ and $f,g \in End(V)$. Since V is a scalar module, then there exists $r,c \in$ Ȑ such that $f(v) = rv$ and $g(v) = cv$ for all $v \in V$. Now, we have $(f \circ g)(v) = rcv \in S$, but S is a Nearly-2-Absorabing sub-module of V, it follows that either $rv \in S + \mathcal{J}(V)$ or $cv \in S + \mathcal{J}(V)$ or $rcV \subseteq S + \mathcal{J}(V)$. So that either $f(v) \in S + \mathcal{J}(V)$ or $g(v) \in S + \mathcal{J}(V)$ or $(f \circ g)(V) \subseteq S + \mathcal{J}(V)$. Thus S is STN-2-Absorabing sub-module of V.

Since cyclic Ȑ-module is scalar, so by proposition(2.12) we obtain the next corollary.

3.12 Corollary (3.12)

Let V be a cyclic Ȑ-module and S be a proper sub-module of V. Then S is a STN-2-Absorabing sub-module of V if and only if S is a Nearly-2-Absorabing sub-module of V.

Since finitely-generated module is scalar, then by proposition(2.13) we obtain the next corollary.

3.13 Corollary (3.13)

Let V be a finitely-generated Ȑ-module and S be a proper sub-module of V. Then S is a STN-2-Absorabing sub-module of V if and only if S is a Nearly-2-Absorabing sub-module of$V$.

The following proposition shows that the inverse image of STN-2-Absorabing sub-module is STN-2-Absorabing.

3.14 Proposition (3.14)

Let $h\colon V \to \bar{V}$ be a small epimorphism, S be a fully invariant STN-2-Absorabing proper sub-module of $\bar{V}$, with $h(\bar{V}) \nsubseteq S$ and $(f \circ g)(\bar{v}) \notin S + \mathcal{J}(\bar{V})$. Then $h^{-1}(S)$ is a STN-2-Absorabing sub-module of V.

3.14.1 Proof

Since S is a proper sub-module of $\bar{V}$, then $h^{-1}(S)$ is a proper sub-module of V. Assume that $(f \circ g)(\bar{v}) \in h^{-1}(S)$, for $f,g \in End(V)$, $\bar{v} \in \bar{V}$, implies that $h((f \circ g)(\bar{v})) \in S$, it follows that $(h \circ f)(g(\bar{v})) \in S$, but S is a STN-2-Absorabing sub-module of $\bar{V}$ and $(f \circ g)(\bar{v}) \notin S + \mathcal{J}(\bar{V})$, then either $h\big(g(\bar{v})\big) \in S + \mathcal{J}(\bar{V})$ or $(h \circ g)(g(\bar{V})) \subseteq S + \mathcal{J}(\bar{V})$. If $h\big(g(\bar{v})\big) \in S + \mathcal{J}(\bar{V})$, then $g(\bar{v}) \in h^{-1}(S + \mathcal{J}(\bar{V}))$, that is $g(\bar{v}) \in h^{-1}(S) + \mathcal{J}(V)$. If $(h \circ g)(g(\bar{V})) \subseteq S +$

$\mathcal{J}(\bar{V})$, then $h(f \circ g)(\bar{V}) \subseteq S + \mathcal{J}(\bar{V})$, thus $(f \circ g)(\bar{V}) \subseteq h^{-1}(S) + \mathcal{J}(V)$. Therefore $h^{-1}(S)$ is a STN-2-Absorabing sub-module of V.

In the following proposition prove that the homomorphic image of STN-2-Absorabing sub-module is STN-2-Absorabing.

3.15 Proposition (3.15)

Let S be a fully invariant STN-2-Absorabing sub-module of an R-module V and $h: V \to \bar{V}$ be a small epimorphism, with $Kvr(h) \subseteq S$. Then $h(S)$ is a STN-2-Absorabing sub-module of $\bar{V}$, where $\bar{V}$ is V- projective module.

3.15.1 Proof

Assume that $(f \circ g)(\bar{v}) \in h(S)$ where $f, g \in End(\bar{V})$, $\bar{v} \in \bar{V}$. Since h is a small epimorphism, then $h(\mathcal{J}(V)) = \mathcal{J}(\bar{V})$ and $\mathcal{J}(V) = h^{-1}(\mathcal{J}(\bar{V}))$. Also, $\bar{V}$ is V-projective module, then there exists $f_1, f_2: \bar{V} \to V$ such that $h \circ f_1 = g$. Now, we have $h \circ f_1 \in End(V)$, also, we have $(f \circ g)(\bar{v}) = (h \circ f_1) \circ (h \circ f_2)(\bar{v}) \in h(S)$, implies that $(f \circ g)(\bar{v}) = h(f_1 \circ h \circ f_2(\bar{v})) \in h(S)$, then there exists nonzero element $v \in S$ such that $h(f_1 \circ h \circ f_2(\bar{v})) = h(v)$, it follows that $h(f_1 \circ h \circ f_2(\bar{v}) - v) = 0$ implise that $f_1 \circ h \circ f_2(\bar{v}) - v \in Kvr(h) \subseteq S$, hence $f_1 \circ h \circ f_2(\bar{v}) \in S$. That is $(f_1 \circ h) \circ (f_2 \circ h)(v) \in S$. But S is a STN-2-Absorabing sub-module of V, then either $(f_1 \circ h)(v) \in S + \mathcal{J}(V)$ or $(f_2 \circ h)(v) \in S + \mathcal{J}(V)$ or $(f_1 \circ h) \circ (f_2 \circ h)(V) \subseteq S + \mathcal{J}(V)$. Thus either $f_1(\bar{v}) \in S + \mathcal{J}(\bar{V})$ or $f_2(\bar{v}) \in S + \mathcal{J}(\bar{V})$ or $(f_1 \circ h \circ f_2)(h(V)) \subseteq S + \mathcal{J}(\bar{V})$. Hence $h(S)$ is a STN-2-Absorabing sub-module of $\bar{V}$.

3.16 Proposition (3.16)

Let S and L be proper sub-modules of an R-module V with $L \subseteq S$ and L is fully invariant in V. If $\frac{S}{L}$ is a STN-2-Absorabing sub-module of $\frac{V}{L}$, then S is a STN-2-Absorabing sub-module of V.

3.16.1 Proof

Assume that $(f \circ g)(v) \in S$ where $f, g \in End(V)$, $v \in V$ and let $f_1, g_1: \frac{V}{L} \to \frac{V}{L}$ defined by $f_1(v + L) = f(v) + L$ and $g_1(v + L) = g(v) + L$ for each $v \in V$. Since L is fully invariant, then f_1, g_1 are well-defined. If $v \in S \subseteq V$, then $f_1 \circ g_1(v + L) = f_1(g_1(v) + L) = f(g(v) + L) = (f \circ g)(v) + L \in \frac{S}{L}$. But $\frac{S}{L}$ is a STN-2-Absorabing sub-module of $\frac{V}{L}$, then either $f(v + L) \in \frac{S}{L} + \mathcal{J}(\frac{V}{L})$ or $g(v + L) \in \frac{S}{L} + \mathcal{J}(\frac{V}{L})$ or $(f \circ g)(\frac{V}{L}) \subseteq \frac{S}{L} + \mathcal{J}(\frac{V}{L})$. It follows that either $f(v) + L \in \frac{S}{L} + \mathcal{J}(\frac{V}{L})$ or $g(v) + L \in \frac{S}{L} + \mathcal{J}(\frac{V}{L})$ or $f(g(V)) + L \subseteq \frac{S}{L} + \mathcal{J}(\frac{V}{L})$. Hence either $f(v) \in S + \mathcal{J}(V)$ or $g(v) \in S + \mathcal{J}(V)$ or $(f \circ g)(V) \subseteq S + \mathcal{J}(V)$. Thus S is a STN-2-Absorabing sub-module of V.

3.17 Proposition (3.17)

Let S be STN-2-Absorabing sub-module of an R-module V and L is an V-injective sub-module of V. Then either $L \subseteq S$ or $L \cap S$ is a STN-2-Absorabing sub-module of L.

3.17.1 Proof

Suppose that $L \nsubseteq S$, then $L \cap S$ is a proper sub-module of L. Now, assume that $(f \circ)g(v) \in L \cap S$ where $f, g \in End(L)$ and $v \in V$ with $g(v) \notin L \cap S + \mathcal{J}(L)$, then $g(v) \notin S$. To prove that $f(v) \in L \cap S + \mathcal{J}(L)$ or $(f \circ g)(L) \subseteq L \cap S + \mathcal{J}(L)$.Since L is a V-injective, then there exists $f_1, f_2 : V \to L$ such that $f_1 \circ i = f$ and $f_2 \circ i = g$ where i is the inclusion map from L into V. Clearly $f_1, f_2 \in End(V)$. But $(f \circ g)(v) = (f_1 \circ i) \circ (f_2 \circ i))(v) = (f_1 \circ i \circ f_2)(i(v)) = (f_1 \circ i)(f_2(v)) \in S$. But S is a STN-2-Absorabing sub-module of V and $f_2(v) = g(v) \notin S + \mathcal{J}(V)$, then either $f_1(v) \in S + \mathcal{J}(V)$, then $f_1(v) \in S \cap L + \mathcal{J}(L)$. If $(f_1 \circ f_2)(V) \subseteq S + \mathcal{J}(V)$ as $(f \circ g)(L) = (f_1 \circ i) \circ (f_2 \circ i))(L) = (f_1 \circ i \circ f_2)(i(L)) = (f_1 \circ i)(f_2(L)) \subseteq S + \mathcal{J}(V)$, that is $(f \circ g)(L) \subseteq S + \mathcal{J}(V)$. Also, $(f \circ g)(L) \subseteq L + \mathcal{J}(L)$. Then $(f \circ g)(L) \subseteq L \cap S + \mathcal{J}(L)$. Thus $L \cap S$ is a STN-2-Absorabing sub-module of L.

3.18 Proposition (3.18)

Let K be a maximal and fully invariant sub-module of an R-module V. Then K is a STN-2-Absorabing sub-module of V.

3.18.1 Proof

Assume $(f \circ g)(v) \in K$, where $v \in V$, $f, g \in End(V)$ and $v \notin K$, $f(v) \notin K + \mathcal{J}(V)$. Since K is a maximal and $v \notin K$, then $V = K + \mathrm{R}v$, then $(f \circ g)(V) = (f \circ g)(K) + (f \circ g)(\mathrm{R}v) = (f \circ g)(K) + \mathrm{R}(f \circ g)(v) \subseteq K + \mathcal{J}(V)$ because K is fully invariant. Thus $(f \circ g)(V) \subseteq K + \mathcal{J}(V)$. Hence K is a STN-2-Absorabing sub-module of V.

4 Characterizations of stn-2-absorabing sub-modules in some types of modules

Before we introduce the first result we need to prove this lemma.

4.1 Lemma (4.1)

Let S be a proper sub-module of a multiplication projective R-module V. Then S is Nearly-2-Absorabing sub-module of V if and only if $[S :_{\mathrm{R}} V]$ is Nearly-2-Absorabing ideal of R.

4.1.1 Proof

⇒) Assume that S is Nearly-2-Absorabing sub-module of V, and $abI \subseteq [S :_{\mathrm{R}} V]$ for some ideal I of R and $a, b \in \mathrm{R}$, then $ab(IV) \subseteq S$. But S is Nearly-2-Absorabing sub-module of V, then by definition(2.3) either $a(IV) \subseteq S + \mathcal{J}(V)$ or $b(IV) \subseteq S + \mathcal{J}(V)$ or $abV \subseteq S + \mathcal{J}(V)$. Since V is multiplication, hence$S = [S :_{\mathrm{R}} V]V$, and since V is projective multiplication, then by proposition (2.22) $\mathcal{J}(V) = \mathcal{J}(\mathrm{R})V$. Thus either $aIV \subseteq [S :_{\mathrm{R}} V]V + \mathcal{J}(\mathrm{R})V$ or $bIV \subseteq [S :_{\mathrm{R}} V]V + \mathcal{J}(\mathrm{R})V$ or $abV \subseteq [S :_{\mathrm{R}} V]V + \mathcal{J}(\mathrm{R})V$. Hence either $aI \subseteq [S :_{\mathrm{R}} V] + \mathcal{J}(\mathrm{R})$ or $bI \subseteq [S :_{\mathrm{R}} V] + \mathcal{J}(\mathrm{R})$ or $ab \subseteq [S :_{\mathrm{R}} V] + \mathcal{J}(\mathrm{R}) = \left[[S :_{\mathrm{R}} V] + \mathcal{J}(\mathrm{R}) : \mathrm{R}\right]$.Therefore by definition(2.3) $[S :_{\mathrm{R}} V]$ is a Nearly-2-Absorabing ideal of R.
⇐) Suppose that $[S :_{\mathrm{R}} V]$ is Nearly-2-Absorabing ideal of R, and $aIL \subseteq S$ for $a \in \mathrm{R}$ and some sub-module L of V and for some ideal I of R since V is a multiplication, then $L = \mathcal{J}V$ for some ideal $\mathcal{J}$ of R, that is $aI\mathcal{J}V \subseteq S$, implies that $aI\mathcal{J} \subseteq [S :_{\mathrm{R}} V]$, but $[S :_{\mathrm{R}} V]$ is Nearly-2-

Absorabing ideal of R, then either $a\mathcal{J} \subseteq [S:_{R} V] + \mathcal{J}(R)$ or $I\mathcal{J} \subseteq [S:_{R} V] + \mathcal{J}(R)$ or $aI \subseteq \big[[S:_{R} V] + \mathcal{J}(R):_{R} R\big] = [S:_{R} V] + \mathcal{J}(R)$. Thus either $a\mathcal{J}V \subseteq [S:_{R} V]V + \mathcal{J}(R)V$ or $I\mathcal{J}V \subseteq [S:_{R} V]V + \mathcal{J}(R)V$ or $aIV \subseteq [S:_{R} V]V + \mathcal{J}(R)V$. But V is a projective R-module, then $\mathcal{J}(V) = \mathcal{J}(R)V$, hence either $aL \subseteq S + \mathcal{J}(V)$ or $IL \subseteq S + \mathcal{J}(V)$ or $aI \subseteq [S + \mathcal{J}(V):_{R} V]$. Thus S is Nearly-2-Absorabing sub-module of$V$.

4.2 Proposition (4.2)

Let S be a proper sub-module of a cyclic projective R-module V. Then S is STN-2-Absorabing sub-module of V if and only if $[S:_{R} V]$ is a STN-2-Absorabing ideal of R.

4.2.1 Proof

⇒) Since S is STN-2-Absorabing sub-module of V and V be a cyclic, then by corollary(3.12) S is Nearly-2-Absorabing sub-module of V. But V is a cyclic, then by proposition(2.21) V is a multiplication and by lemma (4.1) $[S:_{R} V]$ is Nearly-2-Absorabing ideal of R, a gain by corollary (3.12) we have $[S:_{R} V]$ is a STN-2-Absorabing ideal of R.

⇐) Now, let $[S:_{R} V]$ is a STN-2-Absorabing ideal of R, then by corollary(3.12) $[S:_{R} V]$ is Nearly-2-Absorabing ideal of R. But V is a cyclic, then by proposition(2.21) V is a multiplication and V is projective. Thus by lemma(4.1) S is Nearly-2-Absorabing sub-module of V, againe by corollary (3.12) we have S is a STN-2-Absorabing sub-module of V.

Again before we introduce the following result we need to proof this lemma.

4.3 Lemma (4.3)

Let S be a proper sub-module of a faithful multiplication R-module V. Then S is a Nearly-2-Absorabing sub-module of V if and only if $[S:_{R} V]$ is Nearly-2-Absorabing ideal of R.

4.3.1 Proof

⇒) Let $abc \in [S:_{R} V]$ for $a, b, c \in R$, then $ab(cV) \subseteq S$. Since S is Nearly-2-Absorabing sub-module of V, then by definition(2.3) either $acV \subseteq S + \mathcal{J}(V)$ or $bcV \subseteq S + \mathcal{J}(V)$ or $abV \subseteq S + \mathcal{J}(V)$. But V is multiplication, then $S = [S:_{R} V]V$ and since V is faithful multiplication, then by proposition (2.24) $\mathcal{J}(V) = \mathcal{J}(R)V$. Thus either $acV \subseteq [S:_{R} V]V + \mathcal{J}(R)V$ or $V \subseteq [S:_{R} V]V + \mathcal{J}(R)V$ or $abV \subseteq [S:_{R} V]V + \mathcal{J}(R)V$, it follows that either $ac \in [S:_{R} V] + \mathcal{J}(R)$ or $bc \in [S:_{R} V] + \mathcal{J}(R)$ or $ab \in [S:_{R} V] + \mathcal{J}(R) = \big[[S:_{R} V] + \mathcal{J}(R):_{R} R\big]$. Hence $[S:_{R} V]$ is Nearly-2-Absorabing ideal of R.

⇐) Suppose that $[S:_{R} V]$ is Nearly-2-Absorabing ideal of R, and $rcL \subseteq S$ for some sub-module L of V and $r, c \in R$, since V is a multiplication, then $L = \mathcal{J}V$, for some ideal $\mathcal{J}$ of R, that is $rc\mathcal{J}V \subseteq S$, implies that $r\mathcal{J}V \subseteq [S:_{R} V]$, but $[S:_{R} V]$ is Nearly-2-Absorabing ideal of R, then either $r\mathcal{J} \subseteq [S:_{R} V] + \mathcal{J}(R)$ or $c\mathcal{J} \subseteq [S:_{R} V] + \mathcal{J}(R)$ or $rc \in \big[[S:_{R} V] + \mathcal{J}(R):_{R} R\big] = [S:_{R} V] + \mathcal{J}(R)$. Thus either $r\mathcal{J}V \subseteq [S:_{R} V]V + \mathcal{J}(R)V$ or $c\mathcal{J}V \subseteq [S:_{R} V]V + \mathcal{J}(R)V$ or $rcV \subseteq [S:_{R} V]V + \mathcal{J}(R)V$. Hence by proposition (2.24) either $rL \subseteq S + \mathcal{J}(V)$ or $cL \subseteq S + \mathcal{J}(V)$ or $rc \in [S + \mathcal{J}(V):_{R} V]$. Thus S is Nearly-2-Absorabing sub-module of V.

4.4 Proposition (4.4)

Let S be a proper sub-module of a cyclic faithful R-module V. Then S is STN-2-Absorabing sub-module of V if and only if $[S:_{R} V]$ is a STN-2-Absorabing ideal of R.

4.4.1 Proof

⇒) Since S is STN-2-Absorabing sub-module of V and V be a cyclic, then by corollary(3.12) S is Nearly-2-Absorabing sub-module of V. But V is cyclic then by proposition(2.21) V is a multiplication and by lemma (4.3) $[S:_{\text{R}} V]$ is Nearly-2-Absorabing ideal of Ɍ, a gain by corollary (3.12) we have $[S:_{\text{R}} V]$ is a STN-2-Absorabing ideal of Ɍ.

⇐) Now, let $[S:_{\text{R}} V]$ is a STN-2-Absorabing ideal of Ɍ, then by corollary(3.12) $[S:_{\text{R}} V]$ is Nearly-2-Absorabing ideal of Ɍ. But V is cyclic, then by proposition(2.21) V is a multiplication and V is faithful. Thus by lemma(4.3) S is Nearly-2-Absorabing sub-module of V, againe by corollary (3.12) we have S is STN-2-Absorabing sub-module of V.

We need to proof the following lemma before we introduced the next result.

4.5 Lemma (4.5)

Let V be a content multiplication Ɍ-module and S be a proper sub-module of V. Then S is Nearly-2-Absorabing sub-module of V if and only if $[S:_{\text{R}} V]$ is Nearly-2-Absorabing ideal of Ɍ.

4.5.1 Proof

⇒) Let $b\mathcal{J}a \subseteq [S:_{\text{R}} V]$ for some ideal $\mathcal{J}$ of Ɍ and $a, b \in$ Ɍ, then $b\mathcal{J}(aV) \subseteq S$. But S is Nearly-2-Absorabing sub-module of V, then by definition(2.3) either $b(aV) \subseteq S + \mathcal{J}(V)$ or $\mathcal{J}(aV) \subseteq S + \mathcal{J}(V)$ or $b\mathcal{J}V \subseteq S + \mathcal{J}(V)$. Since V is multiplication, then $S = [S:_{\text{R}} V]V$, and since V is content, then by proposition(2.26) $\mathcal{J}(V) = \mathcal{J}(\text{Ɍ})V$. Thus either $baV \subseteq [S:_{\text{R}} V]V + \mathcal{J}(\text{Ɍ})V$ or $\mathcal{J}aV \subseteq [S:_{\text{R}} V]V + \mathcal{J}(\text{Ɍ})V$ or $b\mathcal{J}V \subseteq [S:_{\text{R}} V]V + \mathcal{J}(\text{Ɍ})V$. It n by Proposition (1.1.3=1111111111111111111111111111111111111111111111111111111111111111111111 1111111111111111111111111111111111111111111111111111111111111111111111111111 111111111111111111111111111111111111111111111111111111111111111111111follows that either $ba \subseteq [S:_{\text{R}} V] + \mathcal{J}(\text{Ɍ})$ or $\mathcal{J}a \subseteq [S:_{\text{R}} V] + \mathcal{J}(\text{Ɍ})$ or $b\mathcal{J} \subseteq [S:_{\text{R}} V] + \mathcal{J}(\text{Ɍ}) = \left[[S:_{\text{R}} V] + \mathcal{J}(\text{Ɍ}) : \text{Ɍ}\right]$.Hence $[S:_{\text{R}} V]$ is a Nearly-2-Absorabing ideal of Ɍ.

⇐) Assume $[S:_{\text{R}} V]$ is Nearly-2-Absorabing ideal of Ɍ, and $v_1 v_2 L \subseteq S$ for $v_1, v_2 \in V$ and some sub-module L of V, since V is a multiplication, then $L = \mathcal{J}V$, $(v_1) = I_1 V$ and $(v_2) = I_2 V$ for some ideals $\mathcal{J}$, I_1 and I_2 of Ɍ that is $I_1 I_2 \mathcal{J} V \subseteq S$, implies that $I_1 I_2 \mathcal{J} \subseteq [S:_{\text{R}} V]$, but $[S:_{\text{R}} V]$ is Nearly-2-Absorabing ideal of Ɍ, then by definition(2.3) either $I_1 \mathcal{J} \subseteq [S:_{\text{R}} V] + \mathcal{J}(\text{Ɍ})$ or $I_2 \mathcal{J} \subseteq [S:_{\text{R}} V] + \mathcal{J}(\text{Ɍ})$ or $I_1 I_2 \subseteq \left[[S:_{\text{R}} V] + \mathcal{J}(\text{Ɍ}):_{\text{R}} \text{Ɍ}\right] = [S:_{\text{R}} V] + \mathcal{J}(\text{Ɍ})$. Thus either $I_1 \mathcal{J} V \subseteq [S:_{\text{R}} V]V + \mathcal{J}(\text{Ɍ})V$ or $I_2 \mathcal{J} V \subseteq [S:_{\text{R}} V]V + \mathcal{J}(\text{Ɍ})V$ or $I_1 I_2 V \subseteq [S:_{\text{R}} V]V + \mathcal{J}(\text{Ɍ})V$. Hence by proposition (2.26) either $v_1 L \subseteq S + \mathcal{J}(V)$ or $v_2 L \subseteq S + \mathcal{J}(V)$ or $v_1 v_1 \in [S + \mathcal{J}(V):_{\text{R}} V]$. Therefore S is a Nearly-2-Absorabing sub-module of V.

4.6 Proposition (4.6)

Let V be a cyclic content Ɍ-module and S be a proper sub-module of V. Then S is STN-2-Absorabing sub-module of V if and only if $[S:_{\text{R}} V]$ is a STN-2-Absorabing ideal of Ɍ.

4.6.1 Proof

⇒) Since S is STN-2-Absorabing sub-module of V and V be a cyclic, then by corollary(3.12) S is Nearly-2-Absorabing sub-module of V. But V is cyclic then by proposition(2.21) V is a multiplication and by lemma (4.5) $[S:_{\text{R}} V]$ is Nearly-2-Absorabing ideal of Ɍ, a gain by corollary (3.12) we have $[S:_{\text{R}} V]$ is a STN-2-Absorabing ideal of Ɍ.

$\Leftarrow$) Now, let $[S{:}_{\mathrm{R}}V]$ is STN-2-Absorabing ideal of R, then by corollary(3.12) $[S{:}_{\mathrm{R}}V]$ is Nearly-2-Absorabing ideal of R. But V is cyclic then by proposition(2.21) V is a multiplication and V is projective. Thus by lemma(4.5) S is Nearly-2-Absorabing sub-module of V, againe by corollary (3.12) we have S is a STN-2-Absorabing sub-module of V.

We need to prove the following lemma before we introduce the next proposition.

4.7 Lemma (4.7)

Let V be a finitely-generated multiplication projective R-module, and P is an ideal of R with $ann_{\mathrm{R}}(V) \subseteq P$. Then P is a Nearly-2-Absorabing ideal of R if and only if PV is a Nearly-2-Absorabing sub-module of V.

4.7.1 Proof

$\Rightarrow$) Let $K_1K_2K_3 \subseteq PV$, for K_1, K_2 and K_3 are sub-modules of V. Since V is a multiplication, so $K_1 = I_1V$, $K_2 = I_2V$ and $K_3 = I_3V$ for some idealsI_1, I_2, I_3 in R, that is $I_1I_2I_3V \subseteq PV$. But V is a finitely-generated multiplication R-module then by proposition(2.27) $I_1I_2I_3 \subseteq P + ann_{\mathrm{R}}(V)$, but $ann_{\mathrm{R}}(V) \subseteq P$, implies that $P + ann_{\mathrm{R}}(V) = P$, thus $I_1I_2I_3 \subseteq P$. Now, by assumption P is a Nearly-2-Absorabing ideal of R then by proposition(2.3) either $I_1I_3 \subseteq P + \mathcal{J}(\mathrm{R})$ or $I_2I_3 \subseteq P + \mathcal{J}(\mathrm{R})$ or $I_1I_2 \subseteq [P + \mathcal{J}(\mathrm{R}){:}_{\mathrm{R}}\mathrm{R}] = P + \mathcal{J}(\mathrm{R})$, it follows that either $I_1I_3V \subseteq PV + \mathcal{J}(\mathrm{R})V$ or $I_2I_3V \subseteq PV + \mathcal{J}(\mathrm{R})V$ or $I_1I_2V \subseteq PV + \mathcal{J}(\mathrm{R})V$. Since V is a projective then by proposition (2.22) $\mathcal{J}(V) = \mathcal{J}(\mathrm{R})V$, it follows that either $K_1K_3 \subseteq PV + \mathcal{J}(V)$ or $K_2K_3 \subseteq PV + \mathcal{J}(V)$ or $K_1K_2 \subseteq PV + \mathcal{J}(V)$. Hence by definition(2.3) PV is a Nearly-2-Absorabing sub-module of V.

$\Leftarrow$ Let $I_1I_2I_3 \subseteq P$, for I_1, I_2 and I_3 are ideals in R, implies that $I_1I_2(I_3V) \subseteq PV$. But PV is a Nearly-2-Absorabing sub-module of V, then by definition(2.3) either $I_1(I_3V) \subseteq PV + \mathcal{J}(V)$ or $I_2(I_3V) \subseteq PV + \mathcal{J}(V)$ or $I_1I_2V \subseteq PV + \mathcal{J}(V)$. But V is a projective then $\mathcal{J}(V) = \mathcal{J}(\mathrm{R})V$. Thus either $I_1I_3V \subseteq PV + \mathcal{J}(\mathrm{R})V$ or $I_2I_3V \subseteq PV + \mathcal{J}(\mathrm{R})V$or $I_1I_2V \subseteq PV + \mathcal{J}(\mathrm{R})V$, it follows that either $I_1I_3 \subseteq P + \mathcal{J}(\mathrm{R})$ or $I_2I_3 \subseteq P + \mathcal{J}(\mathrm{R})$or $I_1I_2 \subseteq P + \mathcal{J}(\mathrm{R}) \subseteq [P + \mathcal{J}(\mathrm{R}){:}_{\mathrm{R}}\mathrm{R}]$. Hence by definition(2.3) P is Nearly-2-Absorabing ideal of R.

4.8 Proposition (4.8)

Let V be a cyclic projective R-module and P be a proper ideal of R with $ann_{\mathrm{R}}(V) \subseteq P$. Then P is STN-2-Absorabing ideal of R if and only if PV is STN-2-Absorabing sub-module of V.

4.8.1 Proof

$\Rightarrow$) Assume P is STN-2-Absorabing ideal of R, then by proposition(3.9) P is Nearly-2-Absorabing ideal of R. Since V is cyclic then by proposition(2.21) V is a multiplication, also V is cyclic then V is finitely-generated. Now V is finitely-generated multiplication projective R-module and P is Nearly-2-Absorabing ideal of R then by lemma (4.7) PV is Nearly-2-Absorabing sub-module of V. But V is cyclic then by corollary(3.12) PV is STN-2-Absorabing sub-module of V.

$\Leftarrow$) Suppose PV is STN-2-Absorabing sub-module of V, then by proposition(3.9) PV is Nearly-2-Absorabing sub-module of V. Since V is a cyclic, then V is finitely-generated multiplication and V is projective, then by lemma(4.7) P is Nearly-2-Absorabing ideal of R. But V is cyclic then by corollary(3.12) P is a STN-2-Absorabing ideal of R.

Before we introduce the next proposition need to prove the following lemma.

4.9 Lemma (4.9)

Let V be a faithful finitely-generated multiplication Ɍ-module and P is ideal of Ɍ. Then P is Nearly-2-Absorabing ideal of Ɍ if and only if PV is Nearly-2-Absorabing sub-module of V.

4.9.1 Proof

⇒) Let $rcL \subseteq PV$ for any $r, c \in$ Ɍ, L is a sub-module of V. Since V is a multiplication, then $L = IV$ for some ideal I of Ɍ, that is $rcIV \subseteq PV$. Thus by proposition(2.27) we get $rcI \subseteq P + ann(V)$, but V is faithful, it follows $ann(V) = \{0\}$, that is $rcI \subseteq P$. Since P is a Nearly-2-Absorabing ideal of Ɍ, then by definition(2.3) either $rI \subseteq P + \mathcal{J}(Ɍ)$ or $cI \subseteq P + \mathcal{J}(Ɍ)$ or $rc \in [P + \mathcal{J}(Ɍ) :_{Ɍ} Ɍ] = P + \mathcal{J}(Ɍ)$, hence either $rIV \subseteq PV + \mathcal{J}(Ɍ)V$ or $cIV \subseteq PV + \mathcal{J}(Ɍ)V$or $rcV \subseteq PV + \mathcal{J}(Ɍ)V$, hence by proposition(2.24) either $rL \subseteq PV + \mathcal{J}(V)$ or $cL \subseteq PV + \mathcal{J}(V)$ or $rcV \subseteq PV + \mathcal{J}(V)$. Thus PV is a Nearly-2-Absorabing sub-module of V.

⇐) Let $rcI \subseteq P$ for $r, c \in$ Ɍ and I ideal of Ɍ, hence $rc(IV) \subseteq PV$, but PV is a Nearly-2-Absorabing sub-module of V, then either $r(IV) \subseteq PV + \mathcal{J}(V)$ or $c(IV) \subseteq PV + \mathcal{J}(V)$ or $rcV \subseteq PV + \mathcal{J}(V)$. Thus by proposition(2.24) either $rIV \subseteq PV + \mathcal{J}(Ɍ)V$ or $cIV \subseteq PV + \mathcal{J}(Ɍ)V$ or $rcV \subseteq PV + \mathcal{J}(Ɍ)V$, hence either $rI \subseteq P + \mathcal{J}(Ɍ)$ or $cI \subseteq P + \mathcal{J}(Ɍ)$ or $rc \in P + \mathcal{J}(Ɍ) = [P + \mathcal{J}(Ɍ) :_{Ɍ} Ɍ]$. Therefore P is Nearly-2-Absorabing ideal of Ɍ.

4.10 Proposition (4.10)

Let V be a cyclic faithful Ɍ-module and P be a proper ideal of Ɍ. Then P is STN-2-Absorabing ideal of Ɍ if and only if PV is STN-2-Absorabing sub-module of V.

4.10.1 Proof

⇒) Assume P is STN-2-Absorabing ideal of Ɍ, then by proposition(3.9) P is Nearly-2-Absorabing ideal of Ɍ. Since V is cyclic then by proposition(2.21) V is a multiplication, also V is cyclic then V is finitely-generated. Now V is finitely-generated multiplication faithful Ɍ-module and P is Nearly-2-Absorabing ideal of Ɍ then by lemma (4.9) PV is Nearly-2-Absorabing sub-module of V. But V is cyclic then by corollary(3.12) PV is STN-2-Absorabing sub-module of V.

⇐) Suppose PV is STN-2-Absorabing sub-module of V, then by proposition(3.9) PV is Nearly-2-Absorabing sub-module of V. Since V is cyclic so V is finitely-generated multiplication, and V is faithful then by lemma(4.9) P is Nearly-2-Absorabing ideal of Ɍ. But V is cyclic then by corollary(3.12) P is a STN-2-Absorabing ideal of Ɍ.

Now to prove the following lemma before give the next proposition.

4.11 Lemma (4.11)

Let V be a finitely-generated multiplication content Ɍ-module and P is ideal of Ɍ with $ann_{Ɍ}(V) \subseteq P$. Then P is a Nearly-2-Absorabing ideal of Ɍ if andonly if PV is a Nearly-2-Absorabing sub-module of V.

4.11.1 Proof

(⇒) Let $K_1K_2K_3 \subseteq PV$, for K_1, K_2 and K_3 are sub-modules of V. Since V is a multiplication, then $K_1 = I_1V$, $K_2 = I_2V$ and $K_3 = I_3V$ for some ideals I_1, I_2, I_3 of Ɍ, that is $I_1I_2I_3V \subseteq PV$. But V is a finitely-generated multiplication Ɍ-module then by proposition(2.27) $I_1I_2I_3 \subseteq P +$

$ann_{\mathrm{R}}(V)$, but $ann_{\mathrm{R}}(V) \subseteq P$, implies that $P + ann_{\mathrm{R}}(V) = P$, thus $I_1 I_2 I_3 \subseteq P$. Now, by assumption P is a Nearly-2-Absorabing ideal of R then by definition(2.3) either $I_1 I_3 \subseteq P + \mathcal{J}(\mathrm{R})$ or $I_2 I_3 \subseteq P + \mathcal{J}(\mathrm{R})$ or $I_1 I_2 \subseteq [P + \mathcal{J}(\mathrm{R}):_{\mathrm{R}} \mathrm{R}] = P + \mathcal{J}(\mathrm{R})$, it follows that either $I_1 I_3 V \subseteq PV + \mathcal{J}(\mathrm{R})V$ or $I_2 I_3 V \subseteq PV + \mathcal{J}(\mathrm{R})V$ or $I_1 I_2 V \subseteq PV + \mathcal{J}(\mathrm{R})V$. Since V is content then by proposition(2.26) $\mathcal{J}(V) = \mathcal{J}(\mathrm{R})V$, itfollows either $K_1 K_3 \subseteq PV + \mathcal{J}(V)$ or $K_2 K_3 \subseteq PV + \mathcal{J}(V)$ or $K_1 K_2 \subseteq PV + \mathcal{J}(V)$. Hence by definition(2.3) PV is a Nearly-2-Absorabing sub-module of V.

($\Leftarrow$) Let $I_1 I_2 I_3 \subseteq P$, for I_1, I_2 and I_3 are ideals in R, implies that $I_1 I_2 (I_3 V) \subseteq PV$. But PV is a Nearly-2-Absorabing sub-module of V, then by definition(2.3) either $I_1(I_3 V) \subseteq PV + \mathcal{J}(V)$ or $I_2(I_3 V) \subseteq PV + \mathcal{J}(V)$ or $I_1 I_2 V \subseteq PV + \mathcal{J}(V)$. But V is a projective then $\mathcal{J}(V) = \mathcal{J}(\mathrm{R})V$. Thus either $I_1 I_3 V \subseteq PV + \mathcal{J}(\mathrm{R})V$ or $I_2 I_3 V \subseteq PV + \mathcal{J}(\mathrm{R})V$or $I_1 I_2 V \subseteq PV + \mathcal{J}(\mathrm{R})V$, it follows that either $I_1 I_3 \subseteq P + \mathcal{J}(\mathrm{R})$ or $I_2 I_3 \subseteq P + \mathcal{J}(\mathrm{R})$or $I_1 I_2 \subseteq P + \mathcal{J}(\mathrm{R}) \subseteq [P + \mathcal{J}(\mathrm{R}):_{\mathrm{R}} \mathrm{R}]$. Hence by definition(2.3) P is a Nearly-2-Absorabing ideal of R.

4.12 Proposition (4.12)

Let V be a cyclic content R-module and P be a proper ideal of R, with $ann_{\mathrm{R}}(V) \subseteq P$. Then P is STN-2-Absorabing ideal of R if and only if PV is STN-2-Absorabing sub-module of V.

4.12.1 Proof

In the same way of proposition(4.8).

References

1. Abdulrahman A., (2015), “2-Absorabing Modules and semi-2-Absorabing Modules”, M. sc. Thesis, University of Baghdad.
2. Kash, F., (1982), “Modules and Rings”, London Math. Soc. Monographs, New York, Academic Press,p.370.
3. Reem T. A. and Shwkea M. R., (2018) “Nearly 2-Absorabing Sub-modules and Related Concepts”, Tikrit Journal, for Pure.Sci, vol.(2) No.(3),PP.(215-221).
4. Darani A. Y. and Soheilniai. F., (2011), “2-Absorabing and Weakly 2-Absorabing Sub-modules”, Tahi Journal. Math, vol.(9),pp.(577-584).
5. Goodearl K. R., (1976), “Ring Theory”, Marcel Dekker, Inc. New York and Basel.,p.206.
6. Yaseen S. M.,(1993), “F- Regular Modules”, M. SC. Thesis, University of Baghdad.
7. Lam T. Y., (1999), “Lectures on Modules and Rings”, Graduate Texts in Mathematics, 189, springer, New York.
8. Bothaynah N. S., (2004), “ Scalar Reflexive Modules “ , M.Sc. Thesis, University of Baghdad.
9. Abbas M. S., (1990), “On Fully Stable Modules”, Ph. D. Thesis, University of Baghdad.
10. Barnard A., (1981), "Multiplication Modules", Journal of Algebra, Vol .(7), PP.(174–178).
11. Nuha H. H., (1996), “The Radicals of Modules”, M.sc. Thesis, university of Baghdad.

12. Lu C. P., (1989), “M-radical of Sub-modules in Modules” , Math. Japan, Vol. (34), pp. (211-219).
13. Smith P. F., (1986), “Some Remarks of Multiplication Modules”, Arch. Math, vol. (50), pp. (223-225).