Strongly nearly-2-absorbing submodules

Shwkaea M. Rajab1* and Haibat K. Mohammadali1

1Department of Mathematics, College of Computer Science and Mathematics, Tikrit University, Tikrit, Iraq.

Abstract. Let $R$ be a commutative ring with identity and $V$ be a unitary left $R$-module. The concept of Strongly Nearly-2-Absorbing sub-modules as a generalization of Endo-2-Absorbing sub-modules and strong form of Nearly-2-Absorbing sub-modules are introduced in this paper. Many examples, basic properties of this concept are introduced. Furthermore we prove that in class of (scalar, cyclic and finitely-generated) modules the two concepts Nearly-2-Absorbing sub-modules and Strongly Nearly-2-Absorbing sub-modules are equivalent. Moreover we prove that Endo-2-Absorbing and Strongly Nearly-2-Absorbing sub-modules are equivalent in class of (semi simple, regular) modules. Also those concepts are equivalent in class of all modules over a $v$-ring. Finally we prove several characterizations of Strongly Nearly-2-Absorbing sub-modules in some types of modules such as (projective, faithful and content) modules in class of cyclic modules.

1 Introduction

Throw out this paper all rings are commutative with identity, and all modules are unitary left $R$-modules. Endo-2-Absorbing sub-modules are the famous concept to start with, and were first introduced by Harfash in (2015) as a strong of 2-Absorbing sub-modules. In this paper we introduce new generalizations of Endo-2-Absorbing sub-modules which we called Strongly Nearly-2-Absorbing sub-modules. These papers consist of three parts. In part one introduce several well-known definitions and propositions that we needed throw this paper. In part two we give some basic proposition, examples of Strongly Nearly-2-Absorbing sub-modules, and prove that every Endo-2-Absorbing is Strongly Nearly-2-Absorbing sub-modules, but not conversely. Also prove that every Strongly Nearly-2-Absorbing is Nearly-2-Absorbing sub-modules, but not conversely. Many basic properties and propositions of Strongly Nearly-2-Absorbing sub-modules are introduce in this part. In part three dell with introduced several characterizations of Strongly Nearly-2-Absorbing sub-modules in some kinds of modules.

*Corresponding author: ali.shebl@st.tu.edu.iq
2 Basic Concept

2.1 Definition (2.1)[1]
A proper sub-module $S$ of an R-module $V$ is called Endo-2-Absorabing sub-module if whenever $(fog)(v) \in S$, for $f, g \in End(V), v \in V$, then either $f(v) \in S$ or $g(v) \in S$ or $(fog)(V) \subseteq S$.

2.2 Definition (2.2)[2]
The Jacobson radical $\mathcal{J}(V)$ of an R-module $V$ is defined as intersection of all maximal sub-modules of $V$.

2.3 Definition (2.3)[3]
"A proper sub-module $S$ of an R-module $V$ is called Naely-2-Absorabing, if whenever $rcv \in S$, for $r, c \in R, v \in V$, implies that either $rv \in S + \mathcal{J}(V)$ or $cv \in S + \mathcal{J}(V)$ or $reV \subseteq S + \mathcal{J}(V)$. And an ideal $I$ of a ring $R$ is called Naely-2-Absorabing ideal of $R$ if $I$ is an Naely-2-Absorabing sub-module of an R-module $R$".

2.4 Definition (2.4)[4]
A proper sub-module $S$ of an R-module $V$ is called 2-Absorabing, if whenever $rcv \in S$, for $r, c \in R, v \in V$, implies that either $rv \in S$ or $cv \in S$ or $rcV \subseteq S$.

2.5 Definition (2.5)[5]
An R-module $V$ is called semi simple if every sub-module of $V$ is a direct summand of $V$. Equivalently $V$ is semi simple if and only if $\mathcal{J}(V) = 0$.

2.6 Proposition (2.6)[2, Exercise (12)]
If $S$ is a sub-module of an R-module $V$, with $S$ is a direct summand of $V$, then $\mathcal{J}\left(\frac{V}{S}\right) = \frac{\mathcal{J}(V) + S}{S}$.

2.7 Definition (2.7)[6]
An R-module $V$ is called regular if $R_{an\pi(V)}$ is regular $\forall v \in V$.

2.8 Proposition (2.8)[6, proposition (3.9)]
Let $V$ be a regular R.-module then $\mathcal{J}(V) = 0$.

2.9 Definition (2.9)[7]
A ring $R$ is v-ring if for any R-module $V$, then $\mathcal{J}(V) = 0$. 
2.10 Proposition (2.10)[2, proposition (9.1.4)(b)]

If $S$ is a sub-module of an $R$-module $V$, with $J\left(\frac{V}{S}\right) = 0$, then $J(V) \subseteq S$.

2.11 Definition (2.11)[8]

An $R$-module $V$ is said to be scalar module if for each $f \in \text{End}(V)$, there exists $r \in R$ such that $f(v) = rv$ for each $v \in V$.

2.12 Proposition (2.12)[8, proposition (1.1.7)]

Every cyclic $R$-module $V$ is scalar module.

2.13 Proposition (2.13)[8, corollary (1.1.11)]

If $V$ is a finitely generated $R$-module, then $V$ is scalar module.

2.14 Definition (2.14)[2]

An $R$-epimorphism $f: V \to V'$ is called small epimorphism if $\ker(f)$ is small sub-module of $V$.

2.15 Definition (2.15)[9]

An $R$-module $V$ is fully stable if each sub-module of $V$ is stable.

2.16 Remark (2.16)[9, p.7]

Every stable sub-module is fully invariant.

2.17 Definition (2.17)[2]

An $R$-module $V$ is called $A$-projective if for each homomorphism $f: V \to W$, where $W$ is $R$-module and each $R$-epimorphism $g: A \to W$ there exists an $R$-homomorphism $h: V \to A$ such that $goh = f$.

2.18 Definition (2.18)[2]

An $R$-module $V$ is called $M$-injective if for each $R$-homomorphism $g: K \to L$, and each $R$-homorphism $f: K \to M$, where $M$ is $R$-module there exists an $R$-homomorphism $h: M \to V$ such that $hof = g$.

2.19 Definition (2.19)[10]

"An $R$-module $V$ is multiplication if every sub-module $S$ of $V$ is of the form $S = IV$ for some ideal $I$ of $R$. Equivalently $V$ is multiplication if $S = [S:R]V$".
2.20 Definition (2.20)[11]

"For any sub-module $S$ and $L$ of a multiplication $R$-module $V$ with $S = IV$, $L = JV$, for some ideals $I$ and $J$ of $R$. The product $SL = IS. JL = IJV$, that is $SL = IL$, in particular $SV = IVV = IV = S".

2.21 Definition (2.21)[10]

If $V$ is cyclic $R$-module, then $V$ is a multiplication.

2.22 Proposition (2.22)[2, Theorem (9.2.1)(9)]

For any projective $R$-module $V$, then $J(V) = J(R)V$.

2.23 Definition (2.23)[2]

An $R$-module $V$ is faithful if $ann(V) = \{r \in R: rv = (0)\} = (0)$.

2.24 Proposition (2.24)[12, Remark p.14]

Let $V$ be faithful multiplication $R$-module, then $J(V) = J(R)V$.

2.25 Definition (2.25)[13]

"An $R$-module $V$ is said content if $(\bigcap_{i \in I} A_i) V = \bigcap_{i \in I} A_i V$, for some family of ideals $A_i$ in $R$".

2.26 Proposition (2.26)[12, proposition (1.11)]

If $V$ is content module then $J(V) = J(R)V$.

2.27 Proposition (2.27)[14, corollary of Theorem(9)]

"Let $V$ be a finitely-generated multiplication $R$-module, $I_1$ and $I_2$ are ideals in $R$. Then $I_1 V \subseteq I_2 V$ if and only if $I_1 \subseteq I_2 + ann_R(V)$".

3 Strongly nearly-2-absorabing sub-modules

In this part of the paper we introduce the definition of Strongly Nearly-2-Absorabing sub-module and give some basic properties, examples of this concept.

3.1 Definition (3.1)

A proper sub-module $S$ of an $R$-module $V$ is called Strongly Nearly-2-Absorabing (for short STN-2-Absorabing) sub-module of $V$, if whenever $(f \circ g)(v) \in S$, for $f, g \in End(V)$ and $v \in V$, implies that either $f(v) \in S + J(V)$ or $g(v) \in S + J(V)$ or $(f \circ g)(V) \subseteq S + J(V)$.
3.2 Proposition (3.2)

Every Endo-2-Absorabing sub-module $S$ of an $R$-module $V$ is STN-2-Absorabing sub-module.

3.2.1 Proof

Let $S$ be Endo-2-Absorabing sub-module of an $R$-module $V$ and let $(f \circ g)(v) \in S$, for $f, g \in \text{End}(V)$ and $v \in V$, then $f(v) \in S \subseteq S + J(V)$ or $g(v) \in S \subseteq S + J(V)$ or $f(v) \in S \subseteq S + J(V)$, $(f \circ g)(V) \in S \subseteq S + J(V)$. Thus $S$ is STN-2-Absorabing sub-module of $V$.

The converse of proposition(3.2) is not true in general, the following example explain that.

3.3 Example (3.3)

Consider the $Z$-module $Z_8$ and sub-module $S = \langle 0 \rangle$, $S$ is STN-2-Absorabing sub-module, because $J(Z_8) = \langle 2 \rangle$ but $S$ is not Endo-2-Absorabing. Since $f, g: Z_8 \to Z_8$ defined by $f(v) = 2v$, $g(v) = 2v$ for all $v \in Z_8$, so that $(f \circ g)(2) \in \langle 0 \rangle$. But $f(2) \notin \langle 0 \rangle$ and $g(2) \notin \langle 0 \rangle$ and $(f \circ g)(Z_8) = \{0, 4\} \notin \langle 0 \rangle$.

The convers of proposition(3.2) satisfied in the following propositions under certain conditions.

3.4 Proposition (3.4)

Let $V$ be a semi simple $R$-module and $S$ be a proper sub-module of $V$. Then $S$ is Endo-2-Absorabing sub-module of $V$ if and only if $S$ is STN-2-Absorabing sub-module of $V$.

3.4.1 Proof

$\Rightarrow$) Directly through the proposition(3.2).

$\Leftarrow$) Assume $(f \circ g)(v) \in S$, where $v \in V$ and $f, g \in \text{End}(V)$. Since $V$ is STN-2-Absorabing sub-module of $V$, then either $f(v) \in S + J(V)$ or $g(v) \in S + J(V)$ or $(f \circ g)(V) \subseteq S + J(V)$. Since $V$ is a semi simple, then $J(V) = 0$. Therefore $f(v) \in S$ or $g(v) \in S$ or $(f \circ g)(V) \subseteq S$. Hence $S$ is Endo-2-Absorabing sub-module of $V$.

3.5 Proposition (3.5)

Let $V$ be a regular $R$-module and $S$ be a proper sub-module of $V$. Then $S$ is Endo-2-Absorabing sub-module of $V$ if and only if $S$ is STN-2-Absorabing sub-module of $V$.

3.5.1 Proof

$\Rightarrow$) Directly through the proposition(3.2).

$\Leftarrow$) Assume $(f \circ g)(v) \in S$, where $v \in V$ and $f, g \in \text{End}(V)$. Since $V$ is STN-2-Absorabing sub-module of $V$, then either $f(v) \in S + J(V)$ or $g(v) \in S + J(V)$ or $(f \circ g)(V) \subseteq S + J(V)$. But $V$ is regular, then by proposition(2.8) $J(V) = 0$. Hence $S$ is Endo-2-Absorabing sub-module of $V$. 
3.6 Proposition (3.6)
Let \( V \) be an \( R \)-module over a v-ring \( R \), and \( S \) be a proper sub-module of \( V \). Then \( S \) is Endo-2-Absorabing sub-module of \( V \) if and only if \( S \) is STN-2-Absorabing sub-module of \( V \).

3.6.1 Proof
\[ \Rightarrow \] Directly through the proposition (3.2).
\[ \Leftarrow \] Since \( V \) is an \( R \)-module over v-ring \( R \), then \( J(V) = 0 \). That is the proof is direct.

3.7 Proposition (3.7)
Let \( V \) be an \( R \)-module, and \( S \) be a proper sub-module of \( V \) with \( J(V) \subseteq S \). Then \( S \) is Endo-2-Absorabing sub-module of \( V \) if and only if \( S \) is STN-2-Absorabing sub-module of \( V \).

3.7.1 Proof
\[ \Rightarrow \] Directly through the proposition (3.2).
\[ \Leftarrow \] Let \((f \circ g)(v) \in S, \text{ for } v \in V \text{ and } f, g \in \text{End}(V)\). Since \( V \) is STN-2-Absorabing sub-module of \( V \), then either \( f(v) \in S + J(V) \) or \( g(v) \in S + J(V) \) or \((f \circ g)(V) \subseteq S + J(V)\). Since \( J(V) \subseteq S \), then \( J(V) + S = S \). Hence the proof is followed.

The following corollary as a direct application of proposition (3.7).

3.8 Corollary (3.8)
Let \( V \) be an \( R \)-module, and \( S \) be a proper sub-module of \( V \) with \( J(V) = 0 \). Then \( S \) is Endo-2-Absorabing sub-module of \( V \) if and only if \( S \) is STN-2-Absorabing sub-module of \( V \).

The following propositions explain the relationships of STN-2-Absorabing sub-modules with Nearly-2-Absorabing sub-modules.

3.9 Proposition (3.9)
Every STN-2-Absorabing sub-module \( S \) of an \( R \)-module \( V \) is Nearly-2-Absorabing sub-module.

3.9.1 Proof
Let \( S \) be STN-2-Absorabing sub-module of an \( R \)-module \( V \) and \( rcv \in S \), for \( r, c \in R \) and \( v \in V \). Let \( f, g \in \text{End}(V) \) defined by \( f(v) = rv, \ g(v) = cv \). Thus \((f \circ g)(v) \in S \). Since \( S \) is STN-2-Absorabing sub-module of \( V \), then either \( f(v) \in S + J(V) \) or \( g(v) \in S + J(V) \) or \((f \circ g)(V) \subseteq S + J(V)\). Thus either \( rv \in S + J(V) \) or \( cv \in S + J(V) \) or \( rc(V) \subseteq S + J(V) \). \( S \) is Nearly-2-Absorabing sub-module of \( V \).

The convers of proposition (3.9) is not true in general as the following example explain that.

3.10 Example (3.10)
Let \( V = Z \oplus Z \), \( R = Z \) and the sub-module \( S = 0 \oplus 7Z \), it is clear that \( S \) is Nearly-2-Absorabing sub-module of \( V \). But \( S \) is not STN-2-Absorabing sub-module to show that.
Let \( f: V \rightarrow V \) define by \( f(v, n) = (n, v) \), \( g: V \rightarrow V \) define by \( g(v, n) = (n, 0) \), for \( v, n \in V \); consider \((f \circ g)(1,7) = f(g(1,7)) = f(7,0) = (0,7) \in S + \mathcal{J}(V) \) and \( g(1,7) = (0,7) \notin S + \mathcal{J}(V) \) and \((f \circ g)(Z \oplus Z) = f(g(Z \oplus Z)) = f(Z, 0) = (0, Z) \notin S + \mathcal{J}(V) \).

The converse of proposition(3.9) is hold in the following results under certain conditions.

3.11 Proposition (3.11)

Let \( V \) be a scalar \( R \)-module and \( S \) be a proper sub-module of \( V \). Then \( S \) is a STN-2-Absorabing sub-module of \( V \) if and only if \( S \) is a Nearly-2-Absorabing sub-module of \( V \).

3.11.1 Proof

\( \Rightarrow \) Directly through the proposition(3.9).

\( \Leftarrow \) Assume that \((f \circ g)(v) \in S, \) where \( v \in V \) and \( f, g \in \text{End}(V) \). Since \( V \) is a scalar module, then there exists \( r, c \in R \) such that \( f(v) = rv \) and \( g(v) = cv \) for all \( v \in V \). Now, we have \((f \circ g)(v) = rcv \in S \), but \( S \) is a Nearly-2-Absorabing sub-module of \( V \), it follows that either \( rcv \in S + \mathcal{J}(V) \) or \( cv \in S + \mathcal{J}(V) \) or \( rcv \in S + \mathcal{J}(V) \). So that either \( f(v) \in S + \mathcal{J}(V) \) or \( g(v) \in S + \mathcal{J}(V) \) or \((f \circ g)(V) \subseteq S + \mathcal{J}(V) \). Thus \( S \) is STN-2-Absorabing sub-module of \( V \).

Since cyclic \( R \)-module is scalar, so by proposition(2.12) we obtain the next corollary.

3.12 Corollary (3.12)

Let \( V \) be a cyclic \( R \)-module and \( S \) be a proper sub-module of \( V \). Then \( S \) is a STN-2-Absorabing sub-module of \( V \) if and only if \( S \) is a Nearly-2-Absorabing sub-module of \( V \).

Since finitely-generated module is scalar, then by proposition(2.13) we obtain the next corollary.

3.13 Corollary (3.13)

Let \( V \) be a finitely-generated \( R \)-module and \( S \) be a proper sub-module of \( V \). Then \( S \) is a STN-2-Absorabing sub-module of \( V \) if and only if \( S \) is a Nearly-2-Absorabing sub-module of \( V \).

The following proposition shows that the inverse image of STN-2-Absorabing sub-module is STN-2-Absorabing.

3.14 Proposition (3.14)

Let \( h: V \rightarrow \bar{V} \) be a small epimorphism, \( S \) be a fully invariant STN-2-Absorabing proper sub-module of \( \bar{V} \), with \( h(\bar{V}) \not\subset S \) and \((f \circ g)(\bar{v}) \not\subset S + \mathcal{J}(\bar{V}) \). Then \( h^{-1}(S) \) is a STN-2-Absorabing sub-module of \( V \).

3.14.1 Proof

Since \( S \) is a proper sub-module of \( \bar{V} \), then \( h^{-1}(S) \) is a proper sub-module of \( V \). Assume that \((f \circ g)(\bar{v}) \in h^{-1}(S) \), for \( f, g \in \text{End}(V), \bar{v} \in \bar{V} \), implies that \( h((f \circ g)(\bar{v})) \in S \), it follows that \((h \circ f)(g(\bar{v})) \in S \), but \( S \) is a STN-2-Absorabing sub-module of \( \bar{V} \) and \((f \circ g)(\bar{v}) \not\subset S + \mathcal{J}(\bar{V}) \), then either \( h(g(\bar{v})) \in S + \mathcal{J}(\bar{V}) \) or \((h \circ g)(g(\bar{V})) \not\subset S + \mathcal{J}(\bar{V}) \). If \( h(g(\bar{V})) \in S + \mathcal{J}(\bar{V}) \), then \( g(\bar{v}) \in h^{-1}(S + \mathcal{J}(\bar{V})) \), that is \( g(\bar{v}) \in h^{-1}(S + \mathcal{J}(\bar{V})) \). If \( (h \circ g)(g(\bar{V})) \not\subset S + \mathcal{J}(\bar{V}) \), then \( g(\bar{v}) \in h^{-1}(S + \mathcal{J}(\bar{V})) \), but \( \ldots \)
\[ J(\bar{V}), \text{then } h(f \circ g)(\bar{v}) \subseteq S + J(V), \text{thus } (f \circ g)(\bar{v}) \subseteq h^{-1}(S) + J(V). \text{ Therefore } h^{-1}(S) \text{ is a STN-2-Absorabing sub-module of } V. \]

In the following proposition prove that the homomorphic image of STN-2-Absorabing sub-module is STN-2-Absorabing.

### 3.15 Proposition (3.15)

Let \( S \) be a fully invariant STN-2-Absorabing sub-module of an R-module \( V \) and \( h: V \rightarrow \bar{V} \) be a small epimorphism, with \( Kvr(h) \subseteq S \). Then \( h(S) \) is a STN-2-Absorabing sub-module of \( \bar{V} \), where \( \bar{V} \) is \( V \)-projective module.

#### 3.15.1 Proof

Assume that \((f \circ g)(\bar{v}) \in h(S)\) where \( f,g \in End(\bar{V}), \ \bar{v} \in \bar{V} \). Since \( h \) is a small epimorphism, \( h(J(V)) = J(V) \) and \( J(V) = h^{-1}(J(\bar{V})) \). Also, \( \bar{V} \) is \( V \)-projective module, then there exists \( f_1, f_2: V \rightarrow V \) such that \( h \circ f_1 = g \). Now, we have \((h \circ f_1)(\bar{v}) = \bar{v} \in \bar{V} \). Hence \( f_1 \circ h (f_2(\bar{v})) \in h(S), \) then there exists nonzero element \( v \in S \) such that \((f_1 \circ h \circ f_2(\bar{v})) = h(v) \), it follows that \((f_1 \circ h \circ f_2(\bar{v}) - v) = 0 \) implise that \( f_1 \circ h \circ f_2(\bar{v}) - v \in Kvr(h) \subseteq S \), hence \( f_1 \circ h \circ f_2(\bar{v}) \in S \). That is \((f_1 \circ h)(f_2(\bar{v})) \in S \). But \( S \) is a STN-2-Absorabing sub-module of \( V \), then either \((f_1 \circ h)(\bar{v}) \in S + J(V) \) or \((f_2 \circ h)(\bar{v}) \in S + J(V) \) or \((f_1 \circ h)(\bar{v}) \in S + J(V) \). Thus either \( f_1(\bar{v}) \in S + J(V) \) or \( f_2(\bar{v}) \in S + J(V) \) or \((f_1 \circ h)(\bar{v}) \in S + J(V) \). Hence \( h(S) \) is a STN-2-Absorabing sub-module of \( \bar{V} \).

### 3.16 Proposition (3.16)

Let \( S \) and \( L \) be proper sub-modules of an R-module \( V \) with \( L \subseteq S \) and \( L \) is fully invariant in \( V \). If \( \frac{S}{L} \) is a STN-2-Absorabing sub-module of \( \frac{V}{L} \), then \( S \) is a STN-2-Absorabing sub-module of \( V \).

#### 3.16.1 Proof

Assume that \((f \circ g)(v) \in S \) where \( f,g \in End(V), \ v \in V \) and let \( f_1, g_1: \frac{V}{L} \rightarrow \frac{V}{L} \) defined by \( f_1(v + L) = f(v) + L \) and \( g_1(v + L) = g(v) + L \) for each \( v \in V \). Since \( L \) is fully invariant, then \( f_1, g_1 \) are well-defined. If \( v \in S \subseteq V \), then \( f_1 \circ g_1(v + L) = f_1(g_1(v) + L) = f(g(v) + L) = (f \circ g)(v) + L \subseteq \frac{S}{L} \). But \( \frac{S}{L} \) is a STN-2-Absorabing sub-module of \( \frac{V}{L} \), then either \( f(v + L) \in \frac{S}{L} + J(\frac{V}{L}) \) or \( g(v + L) \in \frac{S}{L} + J(\frac{V}{L}) \) or \((f \circ g)(\frac{v}{L}) \subseteq \frac{S}{L} + J(\frac{V}{L}) \). It follows that either \( f(v) + L \in \frac{S}{L} + J(\frac{V}{L}) \) or \( g(v) + L \in \frac{S}{L} + J(\frac{V}{L}) \) or \( (f \circ g)(V) + L \subseteq \frac{S}{L} + J(\frac{V}{L}) \). Hence either \( f(v) \in S + J(V) \) or \( g(v) \in S + J(V) \) or \((f \circ g)(V) \subseteq S + J(V) \). Thus \( S \) is a STN-2-Absorabing sub-module of \( V \).

### 3.17 Proposition (3.17)

Let \( S \) be STN-2-Absorabing sub-module of an R-module \( V \) and \( L \) is an \( V \)-injective sub-module of \( V \). Then either \( L \subseteq S \) or \( L \cap S \) is a STN-2-Absorabing sub-module of \( L \).
3.17.1 Proof

Suppose that $L \nsubseteq S$, then $L \cap S$ is a proper sub-module of $L$. Now, assume that $(f \circ g)(v) \in L \cap S$ where $f, g \in \text{End}(L)$ and $v \in V$ with $g(v) \notin L \cap S + J(L)$, then $g(v) \notin S$. To prove that $(f \circ g)(v) \in L \cap S + J(L)$ or $(f \circ g)(L) \subseteq L \cap S + J(L)$. Since $L$ is a $V$-injective, then there exists $f_1, f_2: V \to L$ such that $f_1 \circ i = f$ and $f_2 \circ i = g$ where $i$ is the inclusion map from $L$ into $V$. Clearly $f_1, f_2 \in \text{End}(V)$. But $(f \circ g)(v) = (f_1 \circ i)(f_2 \circ i)(v) = (f_1 \circ i \circ f_2)(i(v)) = (f_1 \circ i)(f_2(v)) \in S$. But $S$ is a STN-2-Absorbing sub-module of $V$ and $f_2(v) = g(v) \notin S + J(V)$, then either $f_1(v) \in S + J(V)$, then $f_1(v) \in S \cap L + J(L)$. If $(f_1 \circ f_2)(V) \subseteq S + J(V)$ as $(f \circ g)(L) = (f_1 \circ i)(f_2 \circ i)(L) = (f_1 \circ i)(f_2(L)) \subseteq S + J(V)$, that is $(f \circ g)(L) \subseteq S + J(V)$. Also, $(f \circ g)(L) \subseteq L + J(L)$. Then $(f \circ g)(L) \subseteq L \cap S + J(L)$. Thus $L \cap S$ is a STN-2-Absorbing sub-module of $L$.

3.18 Proposition (3.18)

Let $K$ be a maximal and fully invariant sub-module of an $R$-module $V$. Then $K$ is a STN-2-Absorbing sub-module of $V$.

3.18.1 Proof

Assume $(f \circ g)(v) \in K$, where $v \in V$, $f, g \in \text{End}(V)$ and $v \notin K$, $f(v) \notin K + J(V)$. Since $K$ is a maximal and $v \notin K$, then $V = K + Rv$, then $(f \circ g)(V) = (f \circ g)(K) + (f \circ g)(Rv) = (f \circ g)(K) + R(f \circ g)(v) \subseteq K + J(V)$ because $K$ is fully invariant. Thus $(f \circ g)(V) \subseteq K + J(V)$. Hence $K$ is a STN-2-Absorbing sub-module of $V$.

4 Characterizations of stn-2-absorbing sub-modules in some types of modules

Before we introduce the first result we need to prove this lemma.

4.1 Lemma (4.1)

Let $S$ be a proper sub-module of a multiplication projective $R$-module $V$. Then $S$ is Nearly-2-Absorbing sub-module of $V$ if and only if $[S; R] V$ is Nearly-2-Absorbing ideal of $R$.

4.1.1 Proof

$\Rightarrow$ Assume that $S$ is Nearly-2-Absorbing sub-module of $V$, and $abI \subseteq [S; R] V$ for some ideal $I$ of $R$ and $a, b \in R$, then $ab(IV) \subseteq S$. But $S$ is Nearly-2-Absorbing sub-module of $V$, then by definition (2.3) either $a(IV) \subseteq S + J(V)$ or $b(IV) \subseteq S + J(V)$ or $abV \subseteq S + J(V)$. Since $V$ is multiplication, hence $S = [S; R] V$, and since $V$ is projective multiplication, then by proposition (2.22) $J(V) = J(R)V$. Thus either $aIV \subseteq [S; R] V + J(R)V$ or $bIV \subseteq [S; R] V + J(R)V$ or $abV \subseteq [S; R] V + J(R)V$. Hence either $aI \subseteq [S; R] V + J(R)$ or $bI \subseteq [S; R] V + J(R)$ or $ab \subseteq [S; R] V + J(R) = ([S; R] V + J(R): R)$. Therefore by definition (2.3) $[S; R] V$ is a Nearly-2-Absorbing ideal of $R$.

$\Leftarrow$ Suppose that $[S; R] V$ is Nearly-2-Absorbing ideal of $R$, and $aIL \subseteq S$ for $a \in R$ and some sub-module $L$ of $V$ and for some ideal $I$ of $R$ since $V$ is a multiplication, then $L = IV$ for some ideal $I$ of $R$, that is $alIV \subseteq S$, implies that $alI \subseteq [S; R] V$, but $[S; R] V$ is Nearly-2-
Absorabing ideal of $R$, then either $aJ \subseteq [S_R V] + J(R)$ or $J \subseteq [S_R V] + J(R)$ or $aJ \subseteq [S_R V] + J(R) + R = [S_R V] + J(R)$. Thus either $aJV \subseteq [S_R V]V + J(R)V$ or $J \subseteq [S_R V]V + J(R)V$ or $aJV \subseteq [S_R V]V + J(R)V$. But $V$ is a projective multiplication, then $J(V) = J(R)V$, hence either $aJ \subseteq S + J(V)$ or $L \subseteq S + J(V)$ or $aJ \subseteq [S + J(V);_R V]$. Thus $S$ is Nearly-2-Absorabing sub-module of $V$.

4.2 Proposition (4.2)

Let $S$ be a proper sub-module of a cyclic projective $R$-module $V$. Then $S$ is STN-2-Absorabing sub-module of $V$ if and only if $[S_R V]$ is a STN-2-Absorabing ideal of $R$.

4.2.1 Proof

$\Rightarrow$) Since $S$ is STN-2-Absorabing sub-module of $V$ and $V$ be a cyclic, then by corollary (3.12) $S$ is Nearly-2-Absorabing sub-module of $V$. But $V$ be a cyclic, then by proposition (2.21) $V$ is a multiplication and by lemma (4.1) $[S_R V]$ is Nearly-2-Absorabing ideal of $R$, a gain by corollary (3.12) we have $[S_R V]$ is a STN-2-Absorabing ideal of $R$.

$\Leftarrow$) Now, let $[S_R V]$ is a STN-2-Absorabing ideal of $R$, then by corollary (3.12) $[S_R V]$ is Nearly-2-Absorabing ideal of $R$. But $V$ be a cyclic, then by proposition (2.21) $V$ is a multiplication and $V$ is projective. Thus by lemma (4.1) $S$ is Nearly-2-Absorabing sub-module of $V$, again by corollary (3.12) we have $S$ is a STN-2-Absorabing sub-module of $V$.

Again before we introduce the following result we need to proof this lemma.

4.3 Lemma (4.3)

Let $S$ be a proper sub-module of a faithful multiplication $R$-module $V$. Then $S$ is a Nearly-2-Absorabing sub-module of $V$ if and only if $[S_R V]$ is Nearly-2-Absorabing ideal of $R$.

4.3.1 Proof

$\Rightarrow$) Let $abc \in [S_R V]$ for $a, b, c \in R$, then $ab(cV) \subseteq S$. Since $S$ is Nearly-2-Absorabing sub-module of $V$, then by definition (2.3) either $acV \subseteq S + J(V)$ or $bcV \subseteq S + J(V)$ or $abV \subseteq S + J(V)$. But $V$ is multiplication, then $S = [S_R V]V$ and since $V$ is faithful multiplication, then by proposition (2.24) $J(V) = J(R)V$. Thus either $acV \subseteq [S_R V]V + J(R)V$ or $V \subseteq [S_R V]V + J(R)V$ or $abV \subseteq [S_R V]V + J(R)V$, it follows that either $ac \in [S_R V] + J(R)$ or $bc \in [S_R V] + J(R)$ or $ab \in [S_R V] + J(R) = [S_R V] + J(R);_R R$. Hence $[S_R V]$ is Nearly-2-Absorabing ideal of $R$.

$\Leftarrow$) Suppose that $[S_R V]$ is Nearly-2-Absorabing ideal of $R$, and $rcL \subseteq S$ for some sub-module $L$ of $V$ and $r, c \in R$, since $V$ is a multiplication, then $L = JV$, for some ideal $J$ of $R$, that is $rcJV \subseteq S$, implies that $rJV \subseteq [S_R V]$, but $[S_R V]$ is Nearly-2-Absorabing ideal of $R$, then either $rJ \subseteq [S_R V] + J(R)$ or $cJ \subseteq [S_R V] + J(R)$ or $rc \in [S_R V] + J(R);_R R = [S_R V] + J(R)$. Thus either $rJV \subseteq [S_R V]V + J(R)V$ or $cJV \subseteq [S_R V]V + J(R)V$ or $rcV \subseteq [S_R V]V + J(R)V$. Hence by proposition (2.24) either $rL \subseteq S + J(V)$ or $cL \subseteq S + J(V)$ or $rc \in [S + J(V);_R V]$. Thus $S$ is Nearly-2-Absorabing sub-module of $V$.

4.4 Proposition (4.4)

Let $S$ be a proper sub-module of a cyclic faithful $R$-module $V$. Then $S$ is STN-2-Absorabing sub-module of $V$ if and only if $[S_R V]$ is a STN-2-Absorabing ideal of $R$. 


4.4.1 Proof

\[ \Rightarrow \) Since \( S \) is STN-2-Absorbing sub-module of \( V \) and \( V \) be a cyclic, then by corollary(3.12) \( S \) is Nearly-2-Absorbing sub-module of \( V \). But \( V \) is cyclic then by proposition(2.21) \( V \) is a multiplication and by lemma (4.3) \( [S:R]V \) is Nearly-2-Absorbing ideal of \( R \), a gain by corollary (3.12) we have \( [S:R]V \) is a STN-2-Absorbing ideal of \( R \).

\[ \Leftarrow \) Now, let \( [S:R]V \) is a STN-2-Absorbing ideal of \( R \), then by corollary(3.12) \( [S:R]V \) is Nearly-2-Absorbing ideal of \( R \). But \( V \) is cyclic, then by proposition(2.21) \( V \) is a multiplication and \( V \) is faithful. Thus by lemma(4.3) \( S \) is Nearly-2-Absorbing sub-module of \( V \), again by corollary (3.12) we have \( S \) is STN-2-Absorbing sub-module of \( V \).

We need to prove the following lemma before we introduced the next result.

4.5 Lemma (4.5)

Let \( V \) be a content multiplication \( R \)-module and \( S \) be a proper sub-module of \( V \). Then \( S \) is Nearly-2-Absorbing sub-module of \( V \) if and only if \([S:R]V\) is Nearly-2-Absorbing ideal of \( R \).

4.5.1 Proof

\[ \Rightarrow \) Let \( b\mathcal{I}a \subseteq [S:R]V \) for some ideal \( \mathcal{I} \) of \( R \) and \( a, b \in R \), then \( b\mathcal{I}(aV) \subseteq S \). But \( S \) is Nearly-2-Absorbing sub-module of \( V \), then by definition(2.3) either \( b(aV) \subseteq S + \mathcal{I}(V) \) or \( \mathcal{I}(aV) \subseteq S + \mathcal{I}(V) \) or \( b\mathcal{I}V \subseteq S + \mathcal{I}(V) \). Since \( V \) is multiplication, then \( S = [S:R]V \), and since \( V \) is content, then by proposition(2.26) \( \mathcal{I}(V) = \mathcal{I}(R)V \). Thus either \( baV \subseteq [S:R]V + \mathcal{I}(R)V \) or \( \mathcal{I}(aV) \subseteq [S:R]V + \mathcal{I}(R)V \) or \( b\mathcal{I}V \subseteq [S:R]V + \mathcal{I}(R)V \). Hence \( [S:R]V \) is a Nearly-2-Absorbing ideal of \( R \).

\[ \Leftarrow \) Assume \([S:R]V\) is Nearly-2-Absorbing ideal of \( R \), and \( v_1v_2L \subseteq S \) for \( v_1, v_2 \in V \) and some sub-module \( L \) of \( V \), since \( V \) is a multiplication, then \( L = \mathcal{I}V, (v_1) = I_2V \) and \( (v_2) = I_2V \) for some ideals \( \mathcal{I}, I_1 \) and \( I_2 \) of \( R \) that is \( I_1I_2V \subseteq S \), implies that \( I_1I_2 \subseteq [S:R]V \), but \([S:R]V\) is Nearly-2-Absorbing ideal of \( V \), then by definition(2.3) either \( I_1 \mathcal{I} \subseteq [S:R]V + \mathcal{I}(R) \) or \( I_2 \mathcal{I} \subseteq [S:R]V + \mathcal{I}(R) \) or \( I_1I_2 \subseteq [S:R]V + \mathcal{I}(R) \). Thus either \( I_1V \subseteq [S:R]V + \mathcal{I}(R)V \) or \( I_2V \subseteq [S:R]V + \mathcal{I}(R)V \) or \( I_1I_2V \subseteq [S:R]V + \mathcal{I}(R) \). Hence by proposition (2.26) either \( v_1L \subseteq S + \mathcal{I}(V) \) or \( v_2L \subseteq S + \mathcal{I}(V) \) or \( v_1v_2 \in [S + \mathcal{I}(V):R]V \). Therefore \( S \) is a Nearly-2-Absorbing sub-module of \( V \).

4.6 Proposition (4.6)

Let \( V \) be a cyclic content \( R \)-module and \( S \) be a proper sub-module of \( V \). Then \( S \) is STN-2-Absorbing sub-module of \( V \) if and only if \([S:R]V\) is a STN-2-Absorbing ideal of \( R \).

4.6.1 Proof

\[ \Rightarrow \) Since \( S \) is STN-2-Absorbing sub-module of \( V \) and \( V \) be a cyclic, then by corollary(3.12) \( S \) is Nearly-2-Absorbing sub-module of \( V \). But \( V \) is cyclic then by proposition(2.21) \( V \) is a multiplication and by lemma (4.5) \([S:R]V\) is Nearly-2-Absorbing ideal of \( R \), a gain by corollary (3.12) we have \([S:R]V\) is a STN-2-Absorbing ideal of \( R \).
\(\iff\) Now, let \([S_R V]\) is STN-2-Absorabing ideal of \(R\), then by corollary(3.12) \([S_R V]\) is Nearly-2-Absorabing ideal of \(R\). But \(V\) is cyclic then by proposition(2.22) \([S_R V]\) is a multiplication and \(V\) is projective. Thus by lemma(4.5) \(S\) is Nearly-2-Absorabing sub-module of \(V\), again by corollary (3.12) we have \(S\) is a STN-2-Absorabing sub-module of \(V\).

We need to prove the following lemma before we introduce the next proposition.

4.7 Lemma (4.7)

Let \(V\) be a finitely-generated multiplication projective \(R\)-module, and \(P\) is an ideal of \(R\) with \(\text{ann}_R(V) \subseteq P\). Then \(P\) is a Nearly-2-Absorabing ideal of \(R\) if and only if \(PV\) is a Nearly-2-Absorabing sub-module of \(V\).

4.7.1 Proof

\(\Rightarrow\) Let \(K_1K_2K_3 \subseteq PV\), for \(K_1\), \(K_2\) and \(K_3\) are sub-modules of \(V\). Since \(V\) is a multiplication, so \(K_1 = I_1V, K_2 = I_2V\) and \(K_3 = I_3V\) for some ideals \(I_1, I_2, I_3\) in \(R\), that is \(I_1I_2I_3V \subseteq PV\). But \(V\) is a finitely-generated multiplication \(R\)-module then by proposition(2.27) \(I_1I_2I_3 \subseteq P + \text{ann}_R(V)\), but \(\text{ann}_R(V) \subseteq P\), implies that \(P + \text{ann}_R(V) = P\), thus \(I_1I_2I_3 \subseteq P\). Now, by assumption \(P\) is a Nearly-2-Absorabing ideal of \(R\) then by proposition(2.3) either \(I_1I_3 \subseteq P + J(R)\) or \(I_2I_3 \subseteq P + J(R)\) or \(I_1I_2 \subseteq [P + J(R); R]\) = \(P + J(R)\), it follows that either \(I_1I_3V \subseteq PV + J(R)V\) or \(I_2I_3V \subseteq PV + J(R)V\) or \(I_1I_2V \subseteq PV + J(R)V\). Since \(V\) is a projective then by proposition (2.22) \(J(V) = J(R)V\), it follows that either \(K_1K_3 \subseteq PV + J(V)\) or \(K_2K_3 \subseteq PV + J(V)\) or \(K_1K_2 \subseteq PV + J(V)\). Hence by definition(2.3) \(PV\) is a Nearly-2-Absorabing sub-module of \(V\).

\(\Leftarrow\) Let \(I_1I_2I_3 \subseteq P\), for \(I_1, I_2\) and \(I_3\) are ideals in \(R\), implies that \(I_1I_2(I_3V) \subseteq PV\). But \(PV\) is a Nearly-2-Absorabing sub-module of \(V\), then by definition(2.3) either \(I_1(I_2V) \subseteq PV + J(V)\) or \(I_2(I_3V) \subseteq PV + J(V)\) or \(I_1I_2V \subseteq PV + J(V)\). But \(V\) is a projective then \(J(V) = J(R)V\). Thus either \(I_1I_3V \subseteq PV + J(R)V\) or \(I_2I_3V \subseteq PV + J(R)V\) or \(I_1I_2V \subseteq PV + J(R)V\), it follows that either \(I_1I_3 \subseteq P + J(R)\) or \(I_2I_3 \subseteq P + J(R)\) or \(I_1I_2 \subseteq P + J(R)\). Hence by definition(2.3) \(P\) is Nearly-2-Absorabing ideal of \(R\).

4.8 Proposition (4.8)

Let \(V\) be a cyclic projective \(R\)-module and \(P\) be a proper ideal of \(R\) with \(\text{ann}_R(V) \subseteq P\). Then \(P\) is STN-2-Absorabing ideal of \(R\) if and only if \(PV\) is STN-2-Absorabing sub-module of \(V\).

4.8.1 Proof

\(\Rightarrow\) Assume \(P\) is STN-2-Absorabing ideal of \(R\), then by proposition(3.9) \(P\) is Nearly-2-Absorabing ideal of \(R\). Since \(V\) is cyclic then by proposition(2.21) \(V\) is a multiplication, also \(V\) is cyclic then \(V\) is finitely-generated. Now \(V\) is finitely-generated multiplication \(R\)-module and \(P\) is Nearly-2-Absorabing ideal of \(R\) then by lemma (4.7) \(PV\) is Nearly-2-Absorabing sub-module of \(V\). But \(V\) is cyclic then by corollary(3.12) \(PV\) is STN-2-Absorabing sub-module of \(V\).

\(\Leftarrow\) Suppose \(PV\) is STN-2-Absorabing sub-module of \(V\), then by proposition(3.9) \(PV\) is Nearly-2-Absorabing sub-module of \(V\). Since \(V\) is a cyclic, then \(V\) is finitely-generated multiplication and \(V\) is projective, then by lemma(4.7) \(P\) is Nearly-2-Absorabing ideal of \(R\). But \(V\) is cyclic then by corollary(3.12) \(P\) is a STN-2-Absorabing ideal of \(R\).

Before we introduce the next proposition need to prove the following lemma.
4.9 Lemma (4.9)

Let $V$ be a faithful finitely-generated multiplication R-module and $P$ be ideal of $R$. Then $P$ is Nearly-2-Absorbing ideal of $R$ if and only if $PV$ is Nearly-2-Absorbing sub-module of $V$.

4.9.1 Proof

$\Rightarrow$) Let $rcL \subseteq PV$ for any $r, c \in R$, $L$ is a sub-module of $V$. Since $V$ is a multiplication, then $L = IV$ for some ideal $I$ of $R$, that is $rcIV \subseteq PV$. Thus by proposition(2.27) we get $rcL \subseteq P + ann(V)$, but $V$ is faithful, it follows $ann(V) = \{0\}$, that is $rcL \subseteq P$. Since $P$ is a Nearly-2-Absorbing ideal of $R$, then by definition(2.3) either $rI \subseteq P + J(R)$ or $cL \subseteq P + J(R)$ or $rc \in [P + J(R):_R R] = P + J(R)$, hence either $rIV \subseteq PV + J(R)V$ or $cIV \subseteq PV + J(R)V$ or $rcV \subseteq PV + J(R)V$, hence by proposition(2.24) either $rL \subseteq PV + J(V)$ or $cL \subseteq PV + J(V)$ or $rc \subseteq PV + J(V)$. Thus $PV$ is a Nearly-2-Absorbing sub-module of $V$.

$\Leftarrow$) Let $rcL \subseteq P$ for $r, c \in R$ and $L$ ideal of $R$, hence $rc(IV) \subseteq PV$, but $PV$ is a Nearly-2-Absorbing sub-module of $V$, then either $r(IV) \subseteq PV + J(V)$ or $c(IV) \subseteq PV + J(V)$ or $rcV \subseteq PV + J(V)$. Thus by proposition(2.24) either $rIV \subseteq PV + J(R)V$ or $cIV \subseteq PV + J(R)V$ or $rc \subseteq PV + J(R)V$, hence either $rI \subseteq P + J(R)$ or $cL \subseteq P + J(R)$ or $rc \subseteq P + J(R)$ or $rc \in P + J(R) = [P + J(R):_R R]$. Therefore $P$ is Nearly-2-Absorbing ideal of $R$.

4.10 Proposition (4.10)

Let $V$ be a cyclic faithful $R$-module and $P$ be a proper ideal of $R$. Then $P$ is STN-2-Absorbing ideal of $R$ if and only if $PV$ is STN-2-Absorbing sub-module of $V$.

4.10.1 Proof

$\Rightarrow$) Assume $P$ is STN-2-Absorbing ideal of $R$, then by proposition(3.9) $P$ is Nearly-2-Absorbing ideal of $R$. Since $V$ is cyclic then by proposition(2.21) $V$ is a multiplication, also $V$ is cyclic then $V$ is finitely-generated. Now $V$ is finitely-generated multiplication faithful $R$-module and $P$ is Nearly-2-Absorbing ideal of $R$ then by lemma (4.9) $PV$ is Nearly-2-Absorbing sub-module of $V$. But $V$ is cyclic then by corollary(3.12) $PV$ is STN-2-Absorbing sub-module of $V$.

$\Leftarrow$) Suppose $PV$ is STN-2-Absorbing sub-module of $V$, then by proposition(3.9) $PV$ is Nearly-2-Absorbing sub-module of $V$. Since $V$ is cyclic so $V$ is finitely-generated multiplication, and $V$ is faithful then by lemma(4.9) $P$ is Nearly-2-Absorbing ideal of $R$. But $V$ is cyclic then by corollary(3.12) $P$ is a STN-2-Absorbing ideal of $R$.

Now to prove the following lemma before give the next proposition.

4.11 Lemma (4.11)

Let $V$ be a finitely-generated multiplication content $R$-module and $P$ is ideal of $R$ with $ann_P(V) \subseteq P$. Then $P$ is a Nearly-2-Absorbing ideal of $R$ if and only if $PV$ is a Nearly-2-Absorbing sub-module of $V$.

4.11.1 Proof

$\Rightarrow$) Let $K_1K_2K_3 \subseteq PV$, for $K_1$, $K_2$ and $K_3$ are sub-modules of $V$. Since $V$ is a multiplication, then $K_1 = I_1V$, $K_2 = I_2V$ and $K_3 = I_3V$ for some ideals $I_1, I_2, I_3$ of $R$, that is $I_1I_2I_3V \subseteq PV$. But $V$ is a finitely-generated multiplication $R$-module then by proposition(2.27) $I_1I_2I_3 \subseteq P +$
ann_R(V), but ann_R(V) ⊆ P, implies that P + ann_R(V) = P, thus I_1I_2I_3 ⊆ P. Now, by assumption P is a Nearly-2-Absorabing ideal of R then by definition(2.3) either I_1I_3 ⊆ P + J(R) or I_2I_3 ⊆ P + J(R) or I_1I_2 ⊆ [P + J(R);_R R] = P + J(R). it follows that either I_1I_3V ⊆ PV + J(R)V or I_2I_3V ⊆ PV + J(R)V or I_1I_2V ⊆ PV + J(R)V. Since V is content then by proposition(2.26) J(V) = J(R)V, it follows either K_4K_3 ⊆ PV + J(V) or K_2K_3 ⊆ PV + J(V) or K_1K_2 ⊆ PV + J(V). Hence by definition(2.3) PV is a Nearly-2-Absorabing sub-module of V.

(⇐) Let I_1I_2I_3 ⊆ P, for I_1, I_2 and I_3 are ideals in R, implies that I_1I_2(I_3V) ⊆ PV. But PV is a Nearly-2-Absorabing sub-module of V, then by definition(2.3) either I_1(I_3V) ⊆ PV + J(V) or I_2(I_3V) ⊆ PV + J(V) or I_1I_2V ⊆ PV + J(V). But V is a projective then J(V) = J(R)V. Thus either I_1I_3V ⊆ PV + J(R)V or I_2I_3V ⊆ PV + J(R)V or I_1I_2V ⊆ PV + J(R)V, it follows that either I_1I_3 ⊆ P + J(R) or I_2I_3 ⊆ P + J(R) or I_1I_2 ⊆ P + J(R) ⊆ [P + J(R);_R R]. Hence by definition(2.3) P is a Nearly-2-Absorabing ideal of R.

4.12 Proposition (4.12)

Let V be a cyclic content R-module and P be a proper ideal of R, with ann_R(V) ⊆ P. Then P is STN-2-Absorabing ideal of R if and only if PV is STN-2-Absorabing sub-module of V.

4.12.1 Proof

In the same way of proposition(4.8).

References
