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Lower bound for m_r (2,41) and related code

Hanan Maysar Sabih Yahya [1*] and *Nada Yassen Kasm* Yahya [1]

[1]Department of Mathematics, College of Education for Pure Sciences, University of Mosul, Mosul, Iraq.

Abstract: A (k, n) - arc is a set of k points of a projective plane such that some n, but no n+1 of them, are collinear. The maximum size of a (k, n) - arc in PG(2 q) is denoted by $m_r(2, q)$. In this paper we found $m_r(2,41)$ for n = (2,3 ,...,40) ,In this work, we construct a complete (k, n)-arcs in the projective space over Galois field GF(41), we construct the complete(k,n+1)-arcs from the complete (k, n)-arcs, where $2 \leq n \leq 42$, by using computer program we added some points of index zero, and found all the complete (k, n)-arcs in PG(2,41) and [k,n,d]_code.

1 Introduction

Let GF(q) be the Galois field of q elements and V(3,q) be the vector space of row vectors in length three whose entries in GF(q). Let PG(2,q)be the corresponding projective plane The points of PG(2,q) are the nonzero vectors of V(3,q) with the rule that X=(x_1, x_2, x_3) and Y=($\lambda y_1, \lambda y_2, \lambda y_3$) are the same point, where λ ЄGF(q)\{0}. Since any nonzero vector has precisely q- 1 nonzero scalar multiples, the number of points in the form (1, x_1, x_2) is $q^2$and the form (1, x_1, 0) is q while the form(0,0,1) is1, so the number of points of PG(2,q) is $\frac{q^3-1}{q-1}$= q^2+q+1. If q-1 point P(X) is the equivalence class of the vector X, then we will say that X is a vector representing P(X). A subs pace of dimension one is a set of points all of them representing vectors that form a subs pace of dimension two of V(3,q), such subspaces are called lines. The number of lines in PG(2,q) is q^2 +q+1. There are q+1 points on every line and any two distinct points lie exactly on one line. There are q+ 1 lines throughout every point and any two distinct lines have exactly one common point.

(Bose,in1947) proved that m_2 (2,q) = q+1 for q odd m_2 (2, q) = q + 2 for q even. (Segre, in the mid of 1950s) proved that for q odd every q+ 1-arc is a conic, for q = 2, q = 4 and q = 8 every q+2 -arc is a conic plus its nucleus, and for q = 16, q = 32, q = 2^h (h ≥ 7),there exists a q + 2-arc other than theconic plus its nucleus. (Barlotti,1956) proved that the first of many results in the attempt to determine the value of m_n(2,q), and this has been proved to be far from simple. The existence and the nonexistence of some (k, n)-arcs in PG(2,17) have been proved by (Daskalov, 2004). The existence and the nonexistence of some(k, n)-arcs in PG(2,31) have been proved by (Najem, 2010). Also The existence and the nonexistence of some(k, n)-arcs in PG(2,37) have been proved by (Khalid, 2013).

*) Corresponding author : hanan.22esp39@student.uomosul.edu.iq

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Basic concepts

1.1 [1,5] Definition

Let V be an (n+1)-dimensional vector space over the field K with origin 0 then consider the equivalence relation on the points of V\{0} whose equivalence classes are the one-dimensional subspaces of V with the origin deleted ;that is if x, y ϵV \{0} And for some basis x=(x_0,.., x_n),

y =(y_0,...,y_n) ,X is equivalent to Y if ,for some t in the field K without the origin 0, we have y_i= tx_i for all i. Then, the set of equivalence classes is the n-dimensional projective space over K and is denoted by PG(n, K) or, if K=GF(q), by PG(n, q). In this paper set n=2,K=GF(41), and we get the finite projective plane PG(2,41).

1.2 [1] Definition

A (k, n)-arc A in PG(2,q) is a set of k points such that some line of the plane meets A in n points, but such that no line meets A in more than n points ,where n ≥ 2.A (k, n)_arc is complete if there is no (k+1,n)-arc containing it.

1.3 [1] Definition

Let K[x_0, ..., x_n] be the polynomial ring inx_0,.., x_n variables over the field K, if fin K[x_0,..., x_n] be homogeneous; it is called form. Let X=(x_0,..., x_n),then f(x)=f(x_0,..., x_n). A subset F of PG(n, K) is a variety(over K) if there exists forms f_1, f_2 ,..., f_r in K[x] , such that F= {p(a) ϵPG(n, K) |f_1(a) = f_2(a) = .. = f_r(a) = 0}=V(f_1, f_2,.., f_r). Here, the points p(a) are points of an F.

1.4 [1] Definition

An i-secant for (k ,n)-arc is a line in PG(2,q) such that it contains exactly i points from this (k, n)-arc.

1.5 [1] Theorem

In PG(2, q), q=-1(mod 4) there exists a complete k-arc with k = (q+5)/2.

1.6 1.6.[8] Definition

Let K be a (k, n)-arc if the order of the plane is q then k≤ 1+ (q+ 1)(n - 1) = qn - q+ n with equality if and only if every line intersects the arc in 0 or n points. Arcs realizing the upper bound are called maximal arcs.

1.7 1.7.[1,8] Definition

Points of index zero are points which belong to PG(2,q) such that it does not belong to (k, n)-arc nor to any n-secant for this (k ,n)-arc

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2 Basic definition in PG(n, k)

In this section ,definitions of projective space, linearly dependent or independent points, a subspace PG(n, q), and projectivity are given.

2.1 [1] Definition

Let be of (n + 1) - dimensional vector space over field K .denoted by V (n+1, k) which is considered as an affine space with origin zero. Consider the equivalence relation on the vectors (points) of V\ {0}whose equivalent classes are the one-dimensional subspaces (lines through the origin) of V with the origin neglected; that is if u, v in V\ {0},then u~ v if there exists in K \{0} such that, u = λ v. Therefore .equivalence class: [v] = {u in V \ {0}: ∃λ ϵ K\ {0},u = λv} is a line through the origin. The set of equivalence classes V\ {0} = {[v]: v in V\ {0}}, is the n - dimensional projective space over the field K and is denoted by PG (n, k) or if K = GF(q), by PG(n, q). The elements of PG (n, k) are called points. If the point P(X) is the equivalence class of the vector X, then one Can that X is a vector representing P(X): in this case, t_x for t in

K_0 = K\{0} also represents P(X).

2.2 [1] Definition: Galois Fields

Let GF(p) = Z/PZz_p. p is prime number and let f(x) be an irreducible polynomial of degree h over GF(p), then, GF(p^h)=GF(p)[x]/(f(x)) = {a_0 + a_1x^1 +...+$a_{h-1}x^{h-1}$: a¡ ϵ GF(p), f(t) = 0}, is called a Galois Field of order q; q=p^h The elements of GF(q) satisfy the equation x = x, and there exists y in GF(q) such that

GF(q) = (0,1, y, y^2,....y^{q-2}: y^{q-1} = 1), y is called a primitive root of GF(q)

2.3 [1] Definition

Let (k_1; n)-arc A is of type (T_1, T_{n-1},...,T_0) and (k_2; n)-arc B of type(S_n, S_{n-1},...,S_0) Then A and B have the same type if and only if T_i, $S_i$for all i. In this case they are projectively equivalent.

2.4 [1] Definition

Let T_i be the total number of the i-secants of a (k; n)--arc A, then the type of A denoted by (T_n, T_{n-1}, T_{n-2},...,T_0)

2.5 [1] Definition

Let f(x) =$x^{r+1} - a_r x^r \ldots - a_0$ be any monic polynomial, then its companion matrix, C(f) is given by the (r+1) x (r+ 1) matrix;

$$c(f) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & & & & \\ . & & & & \\ . & & & & \\ a_0 & a_1 & a_2 & \dots & a_r \end{bmatrix}$$

In particular, when r= 2 therefore;

F(x)=x^3 -a_2x^2 -a_1 x^1-a_0

$$C(f) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{bmatrix}$$

2.6 [1] Definition

A projectivity T which permutes the θ(n) points of PG(n,q) in a single cycle is called acyclic projectivity.

2.7 [1] Definition

1. A (k, n) -arc K is a set of k points, such that there is some n but no (n+ 1) are collinear where $n \geq 2$. When n=2 a (k, 2) -arc is called a k-arc.
2. A (k, n) -arc is complete if, there is no (k+1, n) -arc containing it
3. A line L of PG(2,q) is an i-secant of a (k, n)-arc K if, $| L \cap K | = i$. A 0-secant is called an external line of k-arc, a 1-secant is called unisecant and a 2-secant is called a bisecant.
4.A (k, n)-arc K is maximal arc if it satisfies k= (n-1)q+n.
5. The maximum and smallest size of a complete (k, n) -arc for which a (k,n) -arc K exists in PG(2,q) will be denoted by m_n(2,q) and t_n(2.q) respectively.

2.8 [1,8] Corollary

A (k, n) -arc K is maximal if and only if every line in PG(2.q) is either an n-secant or an external line.

2.9 [1] Lemma

If K is a complete (k, n)-arc, then: $(q+1-n)r_n \geq q^2+q+1-k$ with equality if and only if S_n= 1 for all Q in PG(2,q)\K.

2.10 [1] Definition

Let Q_1 and Q_2 be two points of index zero not on the (k, n)-arc K and let k_1= KU $\{Q_1\}$, k_2= KU $\{Q_2\}$ be two arcs, then Q_1 and Q_2 have the same type if and only if k_1 and k_2 are projectively equivalents with respect to the types of lines.

2.11 [5] Definition

A projective plane over GF(q) is 2-dimensional projective space denoted by PG(2,q) and it has the following properties:

1. The number of points is q^2 +q+ 1.
2. The number of lines is q^2+q+1.
3. Each line contains exactly q +1 points.
4. Each point lies on q+ 1 lines.

3 New results

3.1 [1,4] The cyclic projectivity of PG(2,41)

The plane PG(2,41) contains 1723 points and 1723 lines, every line contains 42 points and every point passes through it 42 lines. It is convenience to use the numbers 0,1,2,3,…,40

the matrix T= $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 7 & 1 & 1 \end{bmatrix}$

is cyclic projecivity which is given right multiplication on the points of PG(2,41).

3.2 [1,4] The points lines of PG (2,41)

Let the point $p_1$be represented by the vector (1,0,0). Then $p_1T^{i-1} = p_i$, i=1,2,...,1723 are the 1723 points of PG(2,41) .Writing i forp_i the vectors of the 1723 points PG(2,41) and Let L_1be the line which contains the lines $\{L_1, L_2, L_3, \ldots, L_{1723}\}$ let $L_1T^{i-1} = L_i$,i=1,2,3,…,1723 are the lines of PG(2,41) are given in

Table 1. The points lines of PG (2,41)

I	p_i	l_i
1	(1,0,0)	1,2,69,124,152,176,229,251,258,269,347,382,441,484,628,654,662,675,907,913,944,946,1004,1016,1054,1142,1183,1208,1240,1256,1270,1341,1351,1356,1360,1402,1405,1425,1481,1517,1544,1561
2	(0,1,0)	2,3,70,125,153,177,230,252,259,270,348,383,442,485,629,655,663,676,908,914,945,947,1005,1017,1055,1143,1184,1209,1241,1257,1271,1342,1352,1357,1361,1403,1406,1426,1482,1518,1545,1562
	. . .	. . .
1722	(6,33,1)	1722,1723,67,122,150,174,227,249,256,267,345,380,439,482,626,652,660,673,905,911,942,944,1002,1014,1052,1140,1181,1206,1238,1254,1268,1339,1349,1354,1358,1400,1403,1423,1479,1515,1542,1559
1723	(40,40,1)	1723,1,68,123,151,175,228,250,257,268,346,381,440,483,627,653,661,674,906,912,943,945,1003,1015,1053,1141,1182,1207,1239,1255,1269,1340,1350,1355,1359,1401,1404,1424,1480,1516,1543,1560

4 Algorithm for constructing complete (k,3)-arc degree 2 for PG(2,41) Using Matlab 2019

1- we prove the unit points 1(1,0,0), 2(0,1,0), 3(0,0,1), 323(1, 1, 1) on the Table of points and planes for PG(2,41)
2- we find T_1 for each lines
3- we take the planes that contain 2-secant and the index zero points Co, if co = 0 then the arc is complete. If $C_0 = 0$ we add some points of index zero. The complete arc = {units points} U {added points}

In the same way we construct the others complete arcs.

Note: We will consider the index points of (k, n)-arc. They are all PG(2,41)space points except points of arc from degree n-1 where $2 \leq n \leq 42$

4.1 Construction of complete arcs of degree n in PG(2,41)

In this section we will construct complete (k, n)-arcs in PG(2,41) where $2 \leq n \leq 42$

4.2 Construction of complete (4,2)-arc degree 2 in PG (2,41)

Let δ={1,2,3,323}, where 1(1,0,0), 2(0,1,0), 3(0,0,1),323(1,1,1) which are called the reference and unit points of PG(2,41), δ is (4,2)-arc, since δ intersects any line at most three points. δ is incomplete arc because there exists points of index zero for it which are:

5,..,17,19,20,21,22,25,,26,27,28,29,30,,31,33,…,60,62,63,64,65,66,67,68,71,72,74,75,76,77,78,

81,…,102,104,105,106,107,108,109,110,112,114,117,119,120,121,123,126,…,141,144,145,146,147,148,149,150,151,154,155,156,157,158,161,162,163,165,166,168,169,170,171,172,173,174,

175,178,179,180,181,182,183,184,185,186,188,198,200,…,228,231,…,239,241,242,244,245,246,247,248,249,250,253,254,255,256,257,260,261,262,263,264,266,267,268,271,…,278,280,…,

296,298,…,,305,307,308,309,310,311,312,314,315,316,317,318,319,320,321,325,326,328,…,

346,349,…,381,384,385,386,387,388,389,392,393,394,395,396,397,399,…,407,409,410,411,412,414,415,416,418,…,440,443,444,447,…,458,460,461,463,…,472,475,476,477,478,479,480,481,483,488,489,490,491,492,493,494,495,496,499,500,…,537,539,…,549,552,553,555,…,571,575,576,577,578,581,…,589,592,…,600,602,603,604,605,606,607,608,610,611,612,613,614,615,616,617,619,…,627,630,631,632,633,634,635,636,638,…,653,656,657,658,659,660,664,665,666,

667,670,671,672,673,674,677,…,702,…,713,715,…,735,737,738,739,740,741,742,744,…,753,

755,756,757,758,759,760,761,764,…,780,783,…,804,807,…,831,833,…,848,850,…,866,868,…,903,905,906,909,910,911,912,915,…,925,927,928,929,930,931,933,…,943,948,951,952,953,954,955,957,…,968,970,971,972,973,974,977,978,979,980,981,982,985,…,995,998,999,1000,1001,1002,1003,1006,1007,1008,1010,1011,1012,1013,1014,1015,1018,…,1030,1032,1033,1034,

1035,1036,1037,1039,…,1048,1050,1051,1052,1053,1056,…,1112,1114,…,1126,1128,…,1138,1140,1141,1144,1145,1146,1148,…,1159,1161,1163,…,1182,1185,…,1207,1210,…,1220,1222,1223,1224,1225,1226,1227,1230,,1231,1232,1233,1236,1237,1238,1239,1242,…,1255,1258

,1259,1260,1261,1262,1263,1269,1272,…,1324,1327,…,1336,1339,1340,1343,…,1350,1353,

1354,1355,1358,1359,1362,…,
1374,1377,…,1391,1393,1394,1395,1396,1397,1399,1400,1401,1404,1407,1409,…,1424,1427,

1428,1430,1432,1433,…,1441,1443,…,1454,1456,1457,1458,1459,1460,1461,1462,1465,…,

1480,1483,1484,1485,1486,1487,1488,1490,…,1500,1502,1503,1506,…,1516,1519,…,1528,

1531,…,1538,1540,1541,1542,1543,1546,…,1560,1563,…,1576,1579,…,1590,1593,…,1626,

1628,…,1661,1664,1665,1666,1667,1669,1670,1671,1674,1675,1676,1679,1680,1683,1684,

1685,1686,1688,1689,1690,1691,1692,1694,…,1722, By adding the points 5,6,7,8,9,13,14,15,16,30,35,88,109,232,305,320 to δ we get a complete (20,2)-arc where β = δ U {5,6,7,8,9,13,14,15,16,30,35,88,109,232,305,320}

β = {1,2,3,5,6,7,8,9,13,14,15,16,30,35,88,109,232,305,320,323}

4.3 Construction of complete arcs from degree 3 in PG(2,41)

The different (k,3)-arcs is constructed by adding to β in each time one point from the remaining 1703 points of PG(2,41) as follows:

β1 =β U {4} , β2 = β U {10} , β3= β U {11}, β4= β U {12} , β5 = β U {17} , ..., β18 = β U {31} ,…, β22= βU {63} ,…, β74=β U {89} ,.., β94 = β U {110} ,…, β216= β U {233} ,…, β288=β U {306} ,…, β302= β U {321} ,…, β304=β U {324} ,…, β1703=β U {1723}

We found the intersections of the arcs when n=3 using Matlab program and then we found the equivalent projective arcs. For the arcs, β1,β2,…,β1703 By the definition (2.3.[1]). We found that there are only 8 projective distinct (21,3)-arcs which are:

I	β_i	T_0	T_1	T_2	T_3
1	4,1,2,3,323,5,6,7,8,9,13,14,15,16,30,35,88,109,232,305,320	1046	477	195	5
3	11, 1,2,3,323,5,6,7,8,9,13,14,15,16,30,35,88,109,232,305,320	1045	480	192	6
4	12,1,2,3,323,5,6,7,8,9,13,14,15,16,30,35,88,109,232,305,320	1047	474	198	4
20	33,1,2,3,323,5,6,7,8,9,13,14,15,16,30,35,88,109,232,305,320	1048	471	201	3
28	42,1,2,3,323,5,6,7,8,9,13,14,15,16,30,35,88,109,232,305,320	1049	468	204	2
48	62,1,2,3,323,5,6,7,8,9,13,14,15,16,30,35,88,109,232,305,320	1044	483	189	7
87	102,1,2,3,323,5,6,7,8,9,13,14,15,16,30,35,88,109,232,305,320	1043	486	186	8
206	222,1,2,3,323,5,6,7,8,9,13,14,15,16,30,35,88,109,232,305,320	1050	465	207	1

β_1 is incomplete since there exists points of index zero which are:

10,11,12,17,18,20,21,22,25,26,27,28,29,31,32,33,34,36,37,39,…,50,52,53,54,56,57,59, 60,61,62,63,64,66,67,69,70,71,73,74,75,76,77,79,81,82,83,84,85,86,87,89,…,108,110,111, 112,113,114,

115,117,…,126,128,129,130,131,132,133,135,…,142,145,…,154,156,157,158,159,161, …,169,

171,…,178,181,…,196,198,…,206,208,209,210,211,212,213,215,…,223,225,…,231,23 3,…,253,255,256,257,258,259,260,262,…,271,273,…,282,284,285,286,287,288,290,…,30 4,306,…,319,

321,325,…,349,351,…,376,379,381,382,383,384,386,389,390,393,…,429,431,…,443,4 47,…,

454,456,…,473,475,…,486,488,…,497,499,501,…,509,511,…,517,519,…,527,529,…, 537,539,…,550,553,554,555,556,557,558,560,561,563,…,572,574,575,576,577,578,579,58 1,582,583,585,586,587,588,589,590,592,…,604,606,…,614,617,…,626,628,629,630,633,6 35,636,638,639,640,641,642,643,647,…,654,656,658,659,660,661,662,663,664,666,667,66 8,670,…,677,679,…,703,705,…,716,718,719,720,721,722,724,725,726,728,729,730,731,7 34,735,737,…,757,759,760,761,762,764,765,766,767,769,…,777,779,780,782,…,800,802, 803,804,805,807,…,816,818,…,826,

828,…,856,858,859,861,…,869,871,…,892,894,…,909,911,912,913,914,915,917,918,9 19,921,…,936,938,…,946,948,951,…,975,977,…,983,985,…,996,998,999,1000,1001,1002 ,1003,1005,

1006,1008,1009,1010,1011,1012,1013,1015,1016,1017,1018,1020,…,1029,1031,…,10 37,1039,

1041,1042,1043,1044,1045,1046,1047,1049,1050,1052,1053,1054,1055,1056,1058,105 9,1060, 1062,…,1099,1102,…,1144,1146,…,1167,1169,…,1185,1187,…,1210,1212,…,1222,1224,1225,1226,1227,1228,1230,1231,1232,1233,1234,1236,1237,1238,1239,1240,1241,1242,1244,1245,

1246,1247,1248,1249,1250,1252,1253,1254,1255,1256,1257,1258,1260,1261,1262,1263,1264,

1265,1267,1269,1270,1271,1272,1274,1276,1277,1278,1279,1280,1281,1282,1284,1285,1286,

1287,1288,1290,1291,1292,1294,1295,1296,1297,1298,1300,…,1319,1321,1323,1324,1325,

1327,1329,1331,1333,1334,1335,1336,1337,1339,1340,1341,1342,1343,1345,1346,1347,1348,

1349,1351,1352,1353,1355,1356,1358,1360,1361,1362,1364,1365,1366,1367,1369,1370,1371,

1372,1373,1374,1375,1377,1378,1379,1381,1382,1383,1384,1385,1386,1387,1388,1389,1391,

1393,1394,1395,1396,1397,1398,1399,1400,1401,1403,1404,1406,1407,1409,…,1427,1429,

1431,…,1439,1441,1442,1443,1444,1445,1447,…,1463,1465,…,1480,1482,1483,1485,…,1504,1506,…,1517,1519,1521,1522,1523,1524,1525,1526,1527,1529,1531,…,1539,1541,1542,1543,

1544,1545,1546,1548,…,1558,1560,1561,1563,1565,1566,1567,1568,1570,…,1576,1577,1579,

1580,1581,1582,1585,…,1591,1593,1595,…,1615,1617,…,1625,1627,1628,1629,1630,1631,

1633,…,1641,1643,1644,1645,1647,…,1655,1657,1658,1659,1660,1661,1662,1664,…,1672,

1674,1675,1676,1677,1679,1680,1681,1683,…,1711,17121,1713,1714,1715,1716,1718,1719,

1720,1721,1722,1723

By adding the points
10,11,12,17,18,25,26,27,28,29,33,39,42,45,50,181,251,270,285,346,679,957,1095

We get the complete (44,3)_arc $Ʈ_1$

$Ʈ_1=\beta_1$ U {10,11,12,17,18,25,26,27,28,29,33,39,42,45,50,181,251,270,285,346,679,957,1095} ,,,,

β_3 is incomplete since there exists points of index zero which are:

4,10,12,17,18,19,21,…,29,31,32,33,34,36,…,52,54,55,58,59,61,63,64,66,67,68,70,72,73,74,75,77,78,79,82,84,86,87,89,…,108,111,112,113,114,115,116,118,120,122,123,124,125,126,127,129,130,131,133,134,135,137,138,139,140,142
…,167,169,170,171,173,174,175,176,178,…,194,196,197,198,200,201,202,203,205,208,210,211,213,214,215,217,218,219,220,222,…,231,233,…,240,242,243,244,245,246,247,249,251,…,272,274,276,…,299,301,302,303,304,306,308,…,319,322,324,…,334,336,338,339,340,341,342,344,…,374,377,378,381,382,383,386,…,405,407,409,410,411,412,413,414,415,417,419,420,422,424,426,428,…,442,444,445,446,448,449,450,451,453,…,466,468,471,473,…,486,488,489,491,493,494,495,496,497,499,500,501,503,504,505,506,508,509,510,511,512,514,515,516,517,518,519,

520,522,523,524,525,527,528,529,531,532,533,536,…,545,547,549,551,552,553,555,556,557,

558,560,…,581,583,585,…,602,604,605,606,608,610,613,…,624,627,630,631,633,635,637,638,639,640,642,644,645,646,648,649,650,651,653,…,720,722,723,724,726,727,728,729,731,…,755,757,758,759,761,762,763,764,767,768,769,770,771,773,…,788,793,…,802,804,806,…,814,816,817,818,820,821,822,823,825,…,856,860,861,864,865,866,868,869,87

0,871,872,873,874,876,…,911,915,…,939,941,943,…,963,965,967,…,1000,1003,1004,1005,1007,1008,1009,1010,1012,

1013,1014,1015,1016,1018,1020,…,1027,1029,1030,1031,1033,1034,1035,1038,1043,1044,

1047,1050,1051,1052,1054,1055,1056,1060,1061,1062,1063,1064,1065,1066,1068,…,1098,

1100,…,1114,1116,…,1128,1130,1131,1132,1133,1134,1136,1138,…,1169,1171,1173,…,1199,1201,…,1209,1211,1212,1213,1214,1216,1217,1218,1220,…,1228,1230,1232,…,1260,1262,

1263,1265,1266,1267,1268,1269,1270,1271,1273,1275,1276,1277,1278,1279,1280,1282,1283,

1286,1288,1289,1292,1293,1294,1295,1297,…,1317,1319,1321,1323,1325,1326,1328,1330,…,

1339,1341,..,1375,1377,1379,1380,1381,1383,1384,1385,1386,1388,1389,1391,1392,1393,1395,1396,1397,1398,1400,1401,1402,1404,…,1415,1417,1419,1421,1422,1423,1424,1425,1426,

1427,1429,1430,1431,1433,1434,1435,1436,1438,1439,1440,1441,1443,1445,1446,1447,1448,

1449,1451,1453,…,1462,1464,1466,…,1515,1517,1518,1519,1521,1522,1523,1524,1526,…,

1556,1558,1559,1560,1562,1563,1564,1565,1567,…,1581,1585,1587,1588,1589,1590,1592,…,

1613,1615,1616,1617,1619,1620,1621,1622,1624,1625,1626,1627,1628,1629,1631,1632,1633,

1635,1636,1637,1638,1640,1641,1642,1643,1645,1646,1647,1649,1650,1652,1654,…,1694,

1696,1698,1699,1700,1702,1704,1705,1707,…,1714,1716,1717,1718,1720,1721,1722,1723

By adding the points

4,10,12,17,18,25,26,27,28,29,33,39,42,45,50,181,251,270,285,346,679,957,1095

We get the complete (44,3)_arc μ_1

μ_1 $=\beta_3$ U{4,10,12,17,18,25,26,27,28,29,33,39,42,45,50,181,251,270,285,346,679,957,1095}

β_4 is incomplete since there exists points of index zero which are:

4,10,11,17,…,29,31,33,34,36,…,53,55,56,59,60,62,…,76,78,79,80,82,83,84,85,86,87,89,90,92,…,108,110,…,123,125,126,128,129,130,131,132,134,135,136,138,…,150,152,153,155,…,167,

170,172,174,…,195,197,198,199,201,…,212,214,215,216,218,…,231,233,…,304,306,…,319,

321,322,324,325,326,327,328,329,330,333,336,…,375,378,379,382,…,398,400,401,403,…,443,445,446,447,449,450,451,452,453,455,456,458,…,481,483,484,486,…,498,500,501,502,504,505,507,508,510,…,526,528,529,530,532,…,550,552,553,554,556,557,558,560,561,563,…,580,582,583,585,586,587,589,590,592,593,594,595,596,597,598,600,601,603,605,606,607,609,…,625,

627,628,629,631,632,634,635,636,638,639,640,641,642,643,645,646,647,649,…,676,678,679,

681,…,711,713,714,716,717,718,719,720,721,723,724,725,727,…,756,758,759,760,762,…,770,772,773,775,…,813,815,818,819,821,…,858,860,861,862,864,…,957,959,960,962,…,983,985,

986,988,989,990,991,993,994,996,997,998,999,1000,1001,1002,1004,1006,1009,…,1028,1030,

1031,1032,1034,1035,1036,1038,1039,1040,1042,…,1049,1051,1052,1053,1055,…,1236,1238,
1239,1241,1242,1244,1245,1247,…,1273,1275,1278,1280,1281,1283,1284,1285,1287,1289,
1290,1291,1293,…,1318,1320,1322,1324,1326,…,1333,1335,1336,1338,…,1345,1347,1348,
1350,…,1378,1380,1381,1382,1385,1386,1388,1389,1390,1392,1393,1394,1396,…,1428,1430,
1431,1432,1434,…,1471,1473,1474,1476,…,1512,1514,1515,1518,1519,1520,1522,…,1537,
1539,1540,1542,…,1557,1559,1560,1561,1563,1564,1565,1566,1567,1568,1569,1571,1572,
1574,…,1582,1584,1585,1588,1590,…,1599,1601,1602,1604,…,1614,1616,1617,1618,1620,…,1630,1632,1633,1634,1636,…,1644,1646,1647,1648,1650,…,1670,1672,1673,1675,1676,1677,
1678,1679,1680,1682,1683,1685,1687,1688,1691,1692,1694,…,1715,1717,1718,1719,1721,
1722,1723

By adding the points

4,10,11,17,18,25,26,27,28,29,33,39,42,45,50,181,251,270,285,346,679,957,1095

We get the complete (44,3)_arc ϑ_1

$\vartheta 1 = \beta_4$ U{4,10,11,17,18,25,26,27,28,29,33,39,42,45,50,181,251,270,285,346,679,957,1095}

β_{20} is incomplete since there exists points of index zero which are:

4,10,11,12,17,…,29,31,32,34,36,37,38,39,40,42,…,52,55,…,87,89,90,91,92,94,…,104,106,107,
108,110,…,142,144,146,…,171,173,…,188,190,…,230,233,…,271,273,…,285,287,288,289,290,291,293,294,295,296,298,299,300,301,302,303,304,306,…,319,321,322,324,326,327,328,330,…,344,346,347,348,349,350,351,354,355,356,357,358,360,…,382,384,…,394,396,…,419,421,…
,429,431,432,434,435,436,437,438,439,441,442,443,444,446,447,448,450,…,474,476,…,490,
492,493,495,…,502,504,…,513,515,516,517,518,519,520,522,523,524,525,526,528,…,561,563,564,565,566,567,568,569,571,…,579,581,582,583,584,585,586,588,…,601,603,604,605,607,608,610,…,618,621,…,632,634,636,…,648,651,…,697,699,…,719,721,…,729,731,732,734,736,737,738,740,…,780,782,783,785,786,787,788,789,790,791,793,…,803,805,806,807,808,809,810,
811,812,815,…,834,836,…,859,861,…,880,882,…,895,897,…,922,924,…,935,937,938,939,941,…,963,965,…,978,980,…,987,989,…,1004,1006,1007,1008,1009,1010,1011,1012,1014,…,1025,1027,…,1040,1042,…,1062,1064,1065,1066,1067,1068,1069,1071,…,1080,1082,…,1102,1105,…,1158,1160,…,1170,1172,…,1193,1195,…,1225,1227,…,1252,1255,1256,1257,1259,1260,
1261,1262,1263,1265,…,1277,1279,…,1294,1298,…,1330,1332,…,1352,1354,1356,1357,1358,1359,1361,1362,1363,1364,1365,1366,1368,1369,1370,1372,…,1404,1406,…,1439,1441,1442,
1443,1444,1445,1446,1447,1448,1450,…,1465,1467,1468,1469,1470,1471,1472,1473,1475,…,
1483,1485,1486,1488,1489,1490,1491,1492,1494,…,1533,1535,…,1542,1544,…,1558,1560,…,1585,1587,1588,1589,1590,1592,…,1606,1608,…,1620,1622,…,1691,1693,…,1701,1703,1704,1705,1706,1708,1709,1710,…,1718,1720,1721,1722,1723

By adding the points

4,10,11,12,17,18,25,26,27,28,29,39,42,45,50,181,251,270,285,346,679,957,1095

We get the complete (44,3)_arc ω_1

$\omega_1=\beta_{20}$ U {4,10,11,12,17,18,25,26,27,28,29,39,42,45,50,181,251,270,285,346,679,957,1095}

β_{28} is incomplete since there exists points of index zero which are:

4,10,11,12,17,…,29,31,32,33,34,36,37,38,39,41,43,…,52,54,…,87,89,…,99,101,…,108,110,111,113,…,130,132,…,149,151,…,165,167,…,224,226,227,228,229,230,231,233,234,235,236,237,

239,…,267,269,…,278,280,…,303,306,…,319,321,322,324,…,335,337,…,351,353,…,365,367,…,411,413,…,436,439,440,441,442,443,444,445,448,449,450,451,453,454,455,457,458,460,…,497,499,500,502,…,520,522,…,576,578,…,612,614,…,639,641,…,656,658,…,690,692,693,694,695,696,698,…,727,729,731,…,787,789,…,799,801,…,819,822,…,837,839,…,887,889,…,925,

927,…,942,944,…,966,968,969,970,972,…,991,993,994,996,…,1023,1025,…,1039,1041,1042,

1043,1044,1045,1046,1047,1049,1050,1051,1052,1053,1055,…,1069,1071,1072,1073,1074,1075,1076,1078,…,1087,1089,…,1124,1126,…,1134,1136,1137,1138,1139,1141,1142,1143,1145,…,1165,1167,…,1185,1187,1188,1190,…,1200,1202,1203,1204,1205,1206,1207,1208,1210,…,1259,1261,1262,1263,1264,1266,…,1300,1302,1304,…,1327,1329,…,1344,1346,…,1446,1448,…,1472,1474,1475,1476,1477,1478,1479,1480,1482,…,1493,1495,…,1507,1510,…,1575,1577,…,1630,1632,…,1658,1660,…,1682,1684,…,1723

By adding the points

4,10,11,12,17,18,25,26,27,28,29,33,39,45,50,181,251,270,285,346,679,957,1095

We get the complete (44,3)_arc Ж1

Ж1 $=\beta_{28}$ U,{4,10,11,12,17,18,25,26,27,28,29,33,39,45,50,181,251,270,285,346,679,957,1095}

β_{48} is incomplete since there exists points of index zero which are:

4,10,12,18,19,21,…,29,31,32,33,34,36,…,45,47,…,58,60,61,63,64,66,…,75,77,79,80,81,83,84,

86,87,89,91,93,94,95,97,99,…,108,110,111,112,113,115,…,124,126,…,137,139,140,143,144,

145,146,148,…,156,158,159,160,161,164,165,168,169,170,171,172,175,176,179,180,181,183,

184,185,186,187,188,190,191,192,193,194,195,197,198,199,200,202,203,205,206,209,210,212,

213,214,215,216,217,219,220,222,223,225,226,228,229,230,233,…,246,248,…,284,286,288,…,302,304,307,…,319,321,322,324,…,338,340,342,343,344,345,346,348,349,351,…,365,368,369,370,371,372,374,375,376,377,386,…,421,423,424,425,426,427,429,…,437,439,…,448,451,453,454,455,457,…,479,481,482,483,484,485,486,487,489,…,503,505,506,508,…,529,534,536,…,

545,548,549,550,551,552,553,554,557,558,560,561,563,564,565,567,568,569,570,571,572,574,

575,577,…,591,593,594,595,596,597,599,601,…,608,610,611,615,616,618,…,630,632,633,635,636,637,639,640,643,644,645,646,647,648,650,654,…,668,670,671,672,673,675,676,677,678,

679,680,682,683,684,685,687,688,689,691,692,693,694,695,696,698,699,700,701,702,703,706,…,720,722,723,724,725,726,728,729,731,…,744,746,…,757,759,760,761,763,764,766,767,768,769,770,771,772,773,776,777,778,779,781,782,783,784,786,789,791,792,793,795,797,799,800, 801,802,804,…,813,815,816,817,818,819,820,822,823,825,…,835,837,838,840,841,842,843,844,846,847,848,849,850,852,853,854,855,856,857,858,860,861,862,863,865,866,868,869,870,871,872,873,874,875,877,878,879,880,881,883,885,886,887,888,889,890,891,893,…,910,912,913,

914,916,…,926,928,929,930,931,933,…,941,943,944,945,946,947,948,949,950,952,953,955,956,957,959,960,961,962,963,964,965,967,968,969,971,972,973,974,975,976,977,980,981,982,983,984,985,987,988,989,990,991,993,994,996,…,1007,1009,1010,1012,1014,…,1028,1030,1031,

1032,1033,1035,1036,1038,1039,1040,1041,1043,1044,1046,…,1054,1056,1057,1059,…,1079,

1081,…,1120,1122,…,1145,1147,…,1156,1160,…,1172,1174,1175,1176,1177,1179,1180,1181,1182,1184,1185,1186,1187,1188,1189,1190,1192,1193,1195,1196,1197,1198,1200,1201,1202,

1203,1205,1206,1209,1210,1212,1213,1214,1215,1216,1218,1219,1221,1222,1223,1224,1225,

1227,…,1247,1249,1251,…,1278,1280,1282,1283,1284,1285,1286,1288,1291,1292,1295,1297,

1299,…,1307,1310,…,1319,1321,1322,1323,1325,1328,1330,1331,1332,1334,1335,1336,1337,

1338,1339,1341,1342,1344,…,1357,1359,1360,1361,1362,1364,…,1383,1385,1388,…,1395,

1397,1398,1400,1401,1402,1403,1404,1405,1406,1407,1409,1410,1411,1412,1413,1414,1416,

1417,1418,1420,1421,1422,1423,1424,1425,1427,1428,1429,1430,1431,1432,1433,1435,1438,

1439,1440,1441,1443,1444,1445,1447,1448,1449,1450,1451,1453,1454,1456,1457,1459,…,

1472,1474,1476,1477,1478,1479,1480,1481,1483,1484,1485,1486,1488,1490,1492,1493,1494,

1495,1496,1497,1498,1500,1501,1502,1503,1505,1506,1507,1508,1509,1510,1511,1513,…,

1521,1523,1524,1526,…,1532,1534,1535,1536,1537,1538,1540,1541,1542,1543,1546,1547,

1548,1550,…,1559,1562,1564,1565,1567,1568,1569,1570,1571,1572,1573,1575,…,1582,1584,

1585,1586,1587,1589,1590,1592,1593,1594,1595,1596,1597,1600,…,1619,1621,1622,1624,…,

1635,1637,1638,1640,1642,1643,1644,1646,1647,1648,1649,1651,1652,1654,1656,1657,1658,

1659,1661,1665,1666,1667,1668,1669,1670,1672,…,1686,1688,…,1705,1707,1708,1710,1711,

1713,1714,1715,1716,1717,1718,1719,1720,1722,1723

By adding the points

4,10,12,18,21,26,27,28,29,33,39,40,45,50,71,132,185,244,262,270,274,338,346,699,907

We get the complete (46,3)_arc Ψ_1

$\Psi_1=\beta_{48}$U{4,10,12,18,21,26,27,28,29,33,39,40,45,50,71,132,185,244,262,270,274,338,346,699,
907}

β_{87} is incomplete since there exists points of index zero which are:

4,10,11,12,18,19,20,22,23,25,26,28,29,32,33,34,36,37,38,39,41,42,44,…,71,73,…,79,81,82,83,

84,85,86,89,90,91,93,94,95,96,98,99,100,101,103,104,105,108,110,113,114,115,116,118,119,

122,…,136,138,139,140,141,142,144,146,147,149,150,152,154,155,157,158,159,160,161,162,

164,165,167,169,170,172,174,175,177,178,179,180,181,182,184,185,187,189,190,191,193,194,

195,197,199,201,…,215,217,…,226,228,229,231,235,236,238,240,241,243,…,256,258,259,260,261,262,264,265,266,267,269,…,277,279,…,288,291,293,294,296,297,298,299,300,302,303,304,306,308,309,310,314,317,318,319,321,324,…,334,337,…,348,350,…,361,363,364,366,…,375,

377,378,379,380,381,382,384,386,…,392,394,…,408,410,…,416,418,…,429,431,…,440,442,…,469,472,473,474,478,480,481,482,483,484,487,…,503,506,507,508,509,510,511,512,514,515,

516,517,518,519,520,522,523,524,525,527,…,537,539,540,541,542,544,…,552,554,555,558,…,567,569,570,571,572,573,575,…,592,594,595,596,597,599,600,601,602,603,604,606,609,…,620,622,623,625,627,…,635,637,638,639,640,641,642,643,644,646,647,648,649,651,…,659,661,

663,664,666,667,669,…,676,678,680,681,682,683,686,…,697,700,702,704,705,706,708,709,710,711,712,714,716,717,718,719,721,722,723,724,726,728,729,730,733,734,736,737,739,740,741,744,745,746,747,748,749,750,751,754,755,756,757,760,761,762,763,765,766,767,768,769,770,

772,…,783,785,…,794,796,798,799,800,801,802,804,805,808,810,811,812,814,815,817,818,819,820,822,823,825,826,827,828,829,830,832,833,834,836,837,838,839,840,841,842,843,845,846,847,848,849,850,852,853,854,857,858,860,861,862,863,864,865,868,870,…,896,898,899,900,

903,904,906,907,908,909,911,913,914,916,918,…,924,926,927,928,929,930,932,933,935,936,

939,940,942,…,951,954,955,956,957,959,960,961,962,964,965,966,967,969,…,992,994,997,998,999,1000,1001,1003,…,1010,1013,1014,1015,1016,1018,1019,1020,1022,1023,1024,1026,1027,1028,1029,1031,1032,1033,1035,1036,1037,1038,1040,…,1058,1060,1061,1062,1064,1067,…,1085,1087,1088,1089,1090,1092,1093,1094,1095,1098,1099,1100,1101,1103,1104,1105,1108,

1109,1110,1113,1114,1116,1117,1118,1119,1120,1122,1125,1126,1127,1129,1130,1132,1133,

1134,1135,1137,1139,1140,1141,1143,1145,1147,1148,1150,1151,1152,1153,1154,1155,1156,

1158,1161,1162,1163,1164,1165,1166,1168,…,1177,1179,1181,1182,1184,1185,1186,1187,1188,1189,1190,1191,1193,1194,1195,1196,1197,1198,1199,1200,1201,1203,1204,1206,1208,1210,1212,1213,1214,1215,1216,1218,1219,1220,1221,1223,1224,1225,1226,1227,1228,1229,1231,

1232,1233,1234,1235,1237,…,1252,1254,1255,1257,…,1271,1273,1274,1275,1277,…,1298,

1300,…,1310,1312,1313,1314,1315,1317,…,1331,1333,1334,1335,1336,1338,1339,1340,1341,

1342,1343,1344,1346,…,1357,1359,…,1367,1369,1370,1371,1372,1373,1374,1375,1377,1378,

1379,1380,1383,1384,1388,1389,1390,1392,1393,1394,1396,1398,…,1411,1414,1416,1417,

1419,1421,1422,1423,1424,1425,1426,1427,1429,1431,…,1452,1454,…,1461,1463,14
64,1465,
1466,1467,1469,1470,1471,1472,1474,1475,1476,1477,1481,1482,1483,1484,1486,148
7,1489,
1491,1492,1493,1494,1495,1496,1497,1498,1502,1503,1504,1505,1506,1507,1509,…,
1522,1524,1525,1526,1527,1529,1530,1531,1532,1533,1534,1535,1537,1539,…,1546,154
8,…,1558,1561,…,1570,1572,1573,1574,1577,…,1589,1591,1592,1593,1594,1595,1597,1
598,1599,1600,1601,1603,1604,1605,1606,1607,1608,1610,1612,1614,1615,1617,…,1625,
1628,1631,1632,1633,
1634,1636,1637,1638,1639,1640,1641,1644,…,1651,1655,1656,1658,…,1666,1668,…,
1676,
1678,1679,1680,1681,1683,1684,1685,1686,1688,1689,1690,1691,1693,1694,1695,169
6,1700,
1701,1702,1703,1704,1705,1706,1708,1710,1711,1712,1713,1715,…,1723

By adding the points

4,10,11,12,18,25,26,28,29,39,45,50,123,130,185,190,259,285,346,352,426,783,903,

We get the complete (44,3)_arc ժ$_1$

ժ$_1$=β_{87}U{4,10,11,12,18,25,26,28,29,39,45,50,123,130,185,190,259,285,346,352,426,78
3,903}

β_{206} is incomplete since there exists points of index zero which are:

4,10,11,12,17,…,29,31,32,33,34,36,…,68,70,…,87,89,90,92,93,94,95,96,97,99,…,108,
110,…,
186,188,…,221,223,…,231,233,…,280,282,…,304,306,…,319,321,322,325,…,467,469
,…,493,
495,496,497,498,499,500,501,503,…,514,516,…,746,748,749,750,751,752,754,…,783,
785,787,…,843,845,…,855,857,…,893,895,…,981,983,…,1022,1024,…,1047,1049,…,107
9,1081,…,
1095,1097,…,1109,1111,…,1180,1182,…,1190,1192,1193,1194,1195,1197,1198,1199,
1201,…,1241,1243,1244,1246,…,1264,1266,…,1320,1322,…,1356,1358,…,1383,1385,…,
1400,1402,…,1563,1566,…,1631,1633,…,1686,1688,…,1714,1716,…,1723

By adding the points

4,10,11,12,18,25,26,27,28,29,33,39,42,45,130,185,246,256,464,853,1222,1364,1372

We get the complete (44,3)_arc €$_1$

€$_1$=β_{206}U{4,10,11,12,18,25,26,27,28,29,33,39,42,45,130,185,246,256,464,853,1222,136
4,1372}

4.4 Construction of complete arcs from degree 4 to degree 42 in PG(2,41) for the Arc Ʈ$_1$

(1) from the complete (44,3)_arc Ʈ$_1$ We get a complete (71,4)_arc Ʈ$_2$ Where
Ʈ$_2$=Ʈ$_1$U{19,20,21,22,32,34,37,38,40,41,47,48,55,56,57,61,115,129,202,233,269,29,418
,425,485,691,991}

(2) from the complete (71,4)_arc Ʈ$_2$ we get a complete (101,5)_arc Ʈ$_3$ Where
Ʈ$_3$=Ʈ$_2$U{23,24,31,36,43,44,49,51,52,54,59,62,63,64,72,78,80,146,162,212,230,263,279,30
3,357,472,488,744,1106,1712}

(3) from the complete (101,5)_arc Ʈ$_3$ we get a complete (132,6)_arc Ʈ$_4$ where
Ʈ$_4$=Ʈ$_3$U{46,53,58,65,66,71,75,77,79,84,87,93,101,102,106,127,159,200,228,259,272,311,4
04,
447,518,697,699,714,921,1118,1531}

(4) from the complete (132,6)_arc Ƭ$_4$ we get a complete (166,7)_arc Ƭ$_5$ where Ƭ$_5$=Ƭ$_4$U{60,67,68,69,70,74,76,81,82,86,98,108,110,114,116,151,163,219,283,310,331,335, 343,
440,452,489, 589,901,902,948,1190,1248,1563,1718}

(5) from the complete (166,7)_arc Ƭ$_5$ we get a complete (201,8)_arc Ƭ$_6$ where Ƭ$_6$=Ƭ$_5$U{73,83,85,89,90,91,92,99,104,111,112,113,118,119,126,128,140,170,176,179,199, 217,
300,319,371,394,430,524,571,941,952,1164,1178,1279,1584}

(6) from the complete (201,8)_arc Ƭ$_6$ We get a complete (236,9)_arc Ƭ$_7$ Where

Ƭ$_7$=Ƭ$_6$U{9495,96,97,100,107,117,124,125,132,133,135,136,142,149,161,204,209,287,2 89,298,
307,313,334,347,369,377,522,710,830,906,997,1431,1436,1592}

(7) from the complete (236,9)_arc Ƭ$_7$ We get a complete (267,10)_arc Ƭ$_8$ Where Ƭ$_8$=Ƭ$_7$U{103,105,120,121,137,138,160,168,173,190,192,201,203,227,238,350,351,389,396 ,428,432,441,449,509,529,535,947,1001,1233,1273,1568}

(8) from the complete (267,10)_arc Ƭ$_8$ We get a complete (308,11)_arc Ƭ$_9$ Where Ƭ$_9$=Ƭ$_8$U{122,123,130,131,134,141,144,145,147,174,178,180,185,207,213,216,246,294,312 ,349,375,378,380,401,417,419,427,435,453,476,483,698,790,915,960,1047,1205,1277,130 4,1318,
1643 }

(9) from the complete (308,11)_arc Ƭ$_9$ We get a complete (341,12)_arc Ƭ$_{10}$ Where

Ƭ$_{10}$=Ƭ$_9$U{139,143,150,152,164,165,166,169,171,175,177,182,183,197,225,274,288,326 ,333,395,398,438,459,460,475,566,845,1040,1166,1286,1351,1360,1384}

(10) from the complete (341,12)_arc Ƭ$_{10}$ We get a complete (381,13)_arc Ƭ$_{11}$ Where Ƭ$_{11}$=Ƭ$_{10}$U{148,153,154,155,172,184,186,191,208,214,220,221,226,236,237,240,245,253,26 1,
286,318,329,370,383,423,431,433,446,506,811,1066,1109,1112,1188,1206,1252,1346, 1370,
1510,1587}

(11) from the complete (381,13)_arc Ƭ$_{11}$ We get a complete (414,14)_arc Ƭ$_{12}$ Where Ƭ$_{12}$=Ƭ$_{11}$U{156,157,158,187,189,194,195,222,224,234,248,249,265,267,275,276,321,364,40 0, 411,412,420,443,451,467,478,480,533,538,569,1310,1539,1588}

(12) from the complete (414,14)_arc Ƭ$_{12}$ We get a complete (454,15)_arc Ƭ$_{13}$ Where

Ƭ$_{13}$=Ƭ$_{12}$U{167,188,193,196,198,206,210,211,215,223,239,254,260,264,273,301,327,339,40 2,
434,445,456,461,468,473,474,481,497,555,923,931,1010,1055,1124,1254,1271,1314,1 324,
1331,1359}

(13) from the complete (454,15)_arc Ƭ$_{13}$ We get a complete (488,16)_arc Ƭ$_{14}$ Where

Ƭ$_{14}$=Ƭ$_{13}$U{205,218,229,231,235,242,255,268,277,282,309,317,325,332,338,366,372,42 6,437, 457,463,493,498,513,730,996,998,1049,1099,1335,1366,1394,1395,1411}

(14) from the complete (488,16)_arc Ƭ$_{14}$ We get a complete (529,17)_arc Ƭ$_{15}$ Where Ƭ$_{15}$=Ƭ$_{14}$U{241,243,244,247,257,266,271,278,281,291,292,295,296,342,365,373,374,385,39 2,
439,470,501,531,539,545,583,629,632,753,772,1067,1097,1119,1170,1362,1379,1387, 1410,
1422,1546,1683}

(15) from the complete (529,17)_arc Ƭ$_{15}$ We get a complete (566,18)_arc Ƭ$_{16}$ Where Ƭ$_{16}$=Ƭ$_{15}$U{250,252,258,280,284,290,308,315,336,344,360,367,368,391,393,408,454,465,48 6, 487,496,500,504,536,548,595,597,659,687,708,717,1140,1172,1348,1418,1544,1611}

(16) from the complete (566,18)_arc $Ʈ_{16}$ We get a complete (604,19)_arc $Ʈ_{17}$ Where
$Ʈ_{17}$=$Ʈ_{16}$U{256,262,293,299,306,314,328,330,337,348,359,361,381,397,403,424,458,495,521,
526,543,559,564,600,638,643,690,692,705,721,778,1032,1154,1250,1388,1541,1667,1692 }

(17) from the complete (604,19)_arc $Ʈ_{17}$ We get a complete (639,20)_arc $Ʈ_{18}$ Where

$Ʈ_{18}$=$Ʈ_{17}$U{302,304,316,322,324,340,353,358,362,376,405,413,415,436,442,466,494,520,528,
572,573,594,598,617,718,764,890,974,1139,1141,1176,1241,1338,1389,1416}

(18) from the complete (639,20)_arc $Ʈ_{18}$ We get a complete (680,21)_arc $Ʈ_{19}$ Where

$Ʈ_{19}$=$Ʈ_{18}$U{341,345,352,354,363,379,387,406,407,410,421,429,462,492,499,512,530,534,565,
581,604,607,608,621,630,652,676,695,707,749,844,951,1015,1064,1111,1145,1296,1419,1567,
1602,1627}

(19) from the complete (680,21)_arc $Ʈ_{19}$ We get a complete (720,22)_arc $Ʈ_{20}$ Where

$Ʈ_{20}$=$Ʈ_{19}$U{355,356,382,384,388,390,399,414,416,444,469,527,532,547,562,567,575,586,612,
614,623,636,639,647,673,678,734,815,1131,1137,1209,1245,1341,1421,1438,1515,1576,1608,
1628,1631}

(20) from the complete (720,22)_arc $Ʈ_{20}$ We get a complete (759,23)_arc $Ʈ_{21}$ Where
$Ʈ_{21}$=$Ʈ_{20}$U{386,409,422,464,477,479,484,490,507,511,537,546,552,553,554,577,578,582,596,
648,662,669,681,696,702,703,706,737,748,756,806,817,899,1160,1259,1270,1319,1425,1616}

(21) from the complete (759,23)_arc $Ʈ_{21}$ We get a complete (801,24)_arc $Ʈ_{22}$ Where
$Ʈ_{22}$=$Ʈ_{21}$U{448,450,455,471,482,503,505,508,510,516,544,557,560,563,591,593,615,619,644,
653,654,719,729,731,738,740,742,743,745,766,781,785,788,804,876,943,1068,1306,1358,
1413,1614,1664}

(22) from the complete (801,24)_arc $Ʈ_{22}$ We get a complete (844,25)_arc $Ʈ_{23}$ Where
Ʈ23=$Ʈ_{22}$U{491,502,514,515,523,549,550,568,576,580,588,590,599,602,605,606,620,625,633,
657,700,746,752,757,771,797,802,803,818,953,1061,1175,1203,1300,1321,1334,1340,1399,
1429,1468,1509,1526,1687 }

(23) from the complete (844,25)_arc $Ʈ_{23}$ We get a complete (885,26)_arc $Ʈ_{24}$ Where
$Ʈ_{24}$=$Ʈ_{23}$U{517,519,525,540,541,542,556,574,609,610,613,618,626,628,645,650,665,686,688,
711,759,776,787,812,814,820,822,1189,1267,1282,1455,1511,1538,1582,1585,1607,1619,1647,1675,1680,1704}

(24) from the complete (885,26)_arc $Ʈ_{24}$ We get a complete (927,27)_arc $Ʈ_{25}$ Where
$Ʈ_{25}$=$Ʈ_{24}$U{551,558,561,570,579,585,587,592,601,611,624,627,649,651,660,664,670,716,725,
727,754,755,762,782,796,798,801,836,842,846,852,900,1004,1220,1265,1274,1289,1349,1424,

1556,1659,1722}

(25) from the complete (927,27)_arc Ʈ$_{25}$ We get a complete (972,28)_arc Ʈ$_{26}$ Where
Ʈ$_{26}$=Ʈ25U{584,603,616,622,634,635,637,641,646,655,658,668,680,683,684,685,713,7
22,724,
732,733,774,783,786,793,805,813,823,833,854,856,862,868,918,925,962,1023,1219,12
51,
1433,1434,1506,1522,1686,1705 }

(26) from the complete (972,28)_arc Ʈ$_{26}$ We get a complete (1015,29)_arc Ʈ$_{27}$ Where
Ʈ$_{27}$=Ʈ$_{26}$U{631,640,642,656,661,667,671,682,689,701,704,728,747,751,758,763,765,76
9,784,
799,808,816,829,831,835,848,850,853,880,893,970,983,985,1202,1258,1343,1364,138
3,1562,
1586,1671,1702,1703}

(27) from the complete (1015,29)_arc Ʈ$_{27}$ We get a complete (1060,30)_arc Ʈ$_{28}$ Where
Ʈ$_{28}$=Ʈ$_{27}$U{663,672,677,693,694,709,712,720,723,760,768,773,779,807,810,824,827,83
7,839,
841,860,863,870,877,879,886,903,908,917,927,949,967,1070,1116,1155,1156,1239,12
66,1313,
1392,1415,1450,1604,1624,1715}

(28) from the complete (1060,30)_arc Ʈ$_{28}$ We get a complete (1103,31)_arc Ʈ$_{29}$ Where

Ʈ$_{29}$=Ʈ$_{28}$U{666,674,675,715,726,735,736,739,750,761,767,777,795,819,821,847,857,866,87
5,
885,892,894,897,898,909,914,934,937,954,971,978,995,1071,1094,1110,1133,1195,12
57,1275,
1458,1478,1639,1716}

(29) from the complete (1103,31)_arc Ʈ$_{29}$ We get a complete (1144,32)_arc Ʈ$_{30}$ Where
Ʈ$_{30}$=Ʈ$_{29}$U{741,770,775,780,789,792,794,800,809,825,851,855,867,883,887,910,916,93
2,938,
940,961,979,987,1008,1009,1021,1024,1029,1050,1079,1083,1179,1215,1256,1305,13
15,1357,
1405,1486,1514,1553}

(30) from the complete (1144,32)_arc Ʈ$_{30}$ We get a complete (1191,33)_arc Ʈ$_{31}$ Where
Ʈ$_{31}$=Ʈ$_{30}$U{791,826,828,832,834,838,843,849,858,859,864,872,873,878,881,891,895,91
3,919,
929,939,955,964,966,1020,1022,1027,1034,1046,1059,1063,1093,1096,1149,1182,120
1,1272,
1408,1430,1469,1610,1623,1635,1679,1682,1689,1699 }

(31) from the complete (1191,33)_arc Ʈ$_{31}$ We get a complete (1235,34)_arc Ʈ$_{32}$ Where
Ʈ$_{32}$=Ʈ$_{31}$U{840,861,865,869,871,874,884,888,889,905,911,920,922,926,930,944,959,97
2,973,
976,986,999,1011,1012,1014,1016,1019,1052,1087,1100,1102,1114,1121,1185,1214,1
218,
1249,1263,1350,1461,1606,1650,1685,1691}

(32) from the complete (1235,34)_arc Ʈ$_{32}$ We get a complete (1283,35)_arc Ʈ$_{33}$ Where
Ʈ$_{33}$=Ʈ$_{32}$U{882,896,904,912,928,935,936,942,945,946,958,968,969,975,980,982,984,99
0,992,
1000,1026,1031,1035,1038,1056,1073,1074,1080,1089,1135,1144,1148,1151,1191,119
4,1207,
1208,1236,1284,1302,1320,1325,1363,1449,1471,1517,1566,1657}

(33) from the complete (1283,35)_arc Ʈ$_{33}$ We get a complete (1330,36)_arc Ʈ$_{34}$ Where
Ʈ$_{34}$=Ʈ$_{33}$U{907,924,933,950,956,963,965,977,981,989,993,1002,1007,1013,1018,1028,1
033,
1037,1039,1043,1045,1048,1054,1058,1062,1091,1098,1108,1120,1123,1126,1158,121
2,1227,
1228,1240,1247,1253,1294,1361,1365,1367,1464,1474,1529,1596,1609 }
(34) from the complete (1330,36)_arc Ʈ$_{34}$ We get a complete (1379,37)_arc Ʈ$_{35}$ Where
Ʈ$_{35}$=Ʈ$_{34}$U{988,994,1003,1006,1030,1036,1042,1044,1053,1057,1069,1072,1075,1076,1
078,
1084,1085,1086,1088,1090,1092,1103,1104,1107,1117,1127,1134,1142,1157,1165,117
4,1204,
1211,1226,1255,1269,1280,1290,1326,1344,1368,1377,1451,1456,1540,1555,1590,162
9,1721}
(35) from the complete (1379,37)_arc Ʈ$_{35}$ We get a complete (1428,38)_arc Ʈ$_{36}$ Where
Ʈ$_{36}$=Ʈ$_{35}$U{1005,1017,1025,1051,1060,1077,1081,1082,1101,1105,1113,1115,1125,1128,11
29,
1130,1132,1138,1152,1161,1163,1167,1180,1186,1187,1193,1198,1199,1225,1232,124
4,1264,
1299,1329,1330,1391,1401,1417,1443,1481,1496,1570,1579,1593,1595,1612,1660,166
1,1684}
(36) from the complete (1428,38)_arc Ʈ$_{36}$ We get a complete (1478,39)_arc Ʈ$_{37}$ Where
Ʈ$_{37}$=Ʈ$_{36}$U{1041,1065,1122,1136,1146,1150,1153,1159,1169,1173,1181,1184,1192,1196,12
13,
1221,1222,1223,1229,1234,1238,1242,1261,1262,1276,1278,1287,1291,1292,1293,129
5,1298,
1308,1312,1332,1339,1352,1356,1373,1381,1420,1423,1440,1523,1524,1601,1605,161
7,1673,
1676 }
(37) from the complete (1478,39)_arc Ʈ$_{37}$ We get a complete (1530,40)_arc Ʈ$_{38}$ Where
Ʈ$_{38}$=Ʈ$_{37}$U{1143,1147,1162,1168,1171,1177,1183,1210,1216,1224,1231,1235,1243,1268,12
81,
1283,1285,1288,1297,1307,1311,1322,1323,1327,1328,1347,1354,1355,1369,1376,138
2,1390,
1396,1397,1400,1403,1414,1437,1439,1454,1466,1476,1492,1495,1507,1519,1534,155
0,1633,
1662,1670,1694 }
(38) from the complete (1530,40)_arc Ʈ$_{38}$ We get a complete (1596,41)_arc Ʈ$_{39}$ Where
Ʈ$_{39}$=Ʈ$_{38}$U{1197,1200,1217,1230,1237,1246,1260,1301,1303,1309,1316,1317,1333,1337,13
42,
1353,1371,1372,1374,1378,1386,1393,1398,1402,1404,1406,1409,1412,1427,1428,144
6,1447,
1452,1457,1459,1462,1463,1465,1475,1477,1480,1493,1500,1504,1505,1516,1520,152
1,1533,
1535,1536,1537,1564,1565,1581,1591,1613,1625,1630,1638,1641,1644,1651,1700,170
1,1723}
(39) from the complete (1596,41)_arc Ʈ$_{39}$ We get a complete (1723,42)_arc Ʈ$_{40}$ Where
Ʈ$_{40}$=Ʈ$_{39}$U{1336,1345,1375,1380,1385,1407,1426,1432,1435,1441,1442,1444,1445,1448,14
53,
1460,1467,1470,1472,1473,1479,1482,1483,1484,1485,1487,1488,1489,1490,1491,149
4,1497,

1498,1499,1501,1502,1503,1508,1512,1513,1518,1525,1527,1528,1530,1532,1542,1543,1545,
1547,1548,1549,1551,1552,1554,1557,1558,1559,1560,1561,1569,1571,1572,1573,1574,1575,
1577,1578,1580,1583,1589,1594,1597,1598,1599,1600,1603,1615,1618,1620,1621,1622,1626,
1632,1634,1636,1637,1640,1642,1645,1646,1648,1649,1652,1653,1654,1655,1656,1658,1663,
1665,1666,1668,1669,1672,1674,1677,1678,1681,1688,1690,1693,1695,1696,1697,1698,1706,
1707,1708,1709,1710,1711,1713,1714,1717,1719,1720 }

In the same method for the remaining sets from order 4 in 42 in PG(2,41) we get A complete _arc { µ, ϑ, ω, Ж ,Ψ, ȡ, € }

5 Results and conclusions

From the previous results the distinct complete (k, n)_arc in PG(2,41) , 2 ≤ n ≤ 42 is as follows :

Table 2. the distinct complete (k, n)_arc in PG(2,41)

A complete Ʈ		A complete µ		A complete ϑ		A complete ω		A complete Ж	
N	K	N	K	n	K	N	K	n	K
3	44	3	44	3	44	3	44	3	44
4	71	4	71	4	71	4	71	4	71
5	101	5	101	5	101	5	101	5	101
6	132	6	132	6	132	6	132	6	132
7	166	7	166	7	166	7	166	7	166
8	201	8	201	8	201	8	201	8	201
9	236	9	236	9	236	9	236	9	236
10	267	10	267	10	267	10	267	10	267
11	308	11	308	11	308	11	308	11	308
12	341	12	341	12	341	12	341	12	341
13	381	13	381	13	381	13	381	13	381
14	414	14	414	14	414	14	414	14	414
15	454	15	454	15	454	15	454	15	454
16	488	16	488	16	488	16	488	16	488
17	529	17	529	17	529	17	529	17	529
18	566	18	566	18	566	18	566	18	566
19	604	19	604	19	604	19	604	19	604
20	639	20	639	20	639	20	639	20	639
21	680	21	680	21	680	21	680	21	680
22	720	22	720	22	720	22	720	22	720
23	759	23	759	23	759	23	759	23	759
24	801	24	801	24	801	24	801	24	801
25	844	25	844	25	844	25	844	25	844
26	885	26	885	26	885	26	885	26	885
27	927	27	927	27	927	27	927	27	927
28	972	28	972	28	972	28	972	28	972
29	1015	29	1015	29	1015	29	1015	29	1015
30	1060	30	1060	30	1060	30	1060	30	1060
31	1103	31	1103	31	1103	31	1103	31	1103
32	1144	32	1144	32	1144	32	1144	32	1144

33	1191	33	1191	33	1191	33	1191	33	1191
34	1235	34	1235	34	1235	34	1235	34	1235
35	1283	35	1283	35	1283	35	1283	35	1283
36	1330	36	1330	36	1330	36	1330	36	1330
37	1379	37	1379	37	1379	37	1379	37	1379
38	1428	38	1428	38	1428	38	1428	38	1428
39	1478	39	1478	39	1478	39	1478	39	1478
40	1530	40	1530	40	1530	40	1530	40	1530
41	1596	41	1596	41	1596	41	1596	41	1596
42	1723	42	1723	42	1723	42	1723	42	1723

A complete Ψ		**A complete ȡ**		**A complete €**	
N	K	N	K	n	K
3	46	3	44	3	44
4	71	4	73	4	71
5	104	5	102	5	103
6	135	6	136	6	134
7	168	7	168	7	166
8	201	8	204	8	199
9	233	9	237	9	232
10	268	10	270	10	266
11	303	11	304	11	301
12	339	12	340	12	339
13	373	13	382	13	375
14	408	14	417	14	408
15	446	15	455	15	447
16	487	16	491	16	486
17	526	17	529	17	523
18	564	18	569	18	563
19	602	19	607	19	603
20	641	20	644	20	643
21	682	21	682	21	681
22	724	22	727	22	725
23	766	23	768	23	762
24	804	24	805	24	803
25	847	25	846	25	843
26	884	26	889	26	888
27	927	27	928	27	926
28	974	28	969	28	973
29	1016	29	1013	29	1018
30	1058	30	1061	30	1066
31	1105	31	1102	31	1105
32	1147	32	1149	32	1151
33	1193	33	1190	33	1191
34	1236	34	1236	34	1237
35	1283	35	1283	35	1286
36	1328	36	1325	36	1329
37	1377	37	1374	37	1375
38	1431	38	1425	38	1427
39	1481	39	1480	39	1481
40	1535	40	1535	40	1535
41	1595	41	1597	41	1593
42	1723	42	1723	42	1723

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5.1 [6,7] Theorem

Let K be a (k, n)_arc in PG(2,q) where q is prime then

If n ≤ (q+1)/2 then m_n(2,q) ≤ (n-1)q+1

If n ≥ (q+3)/2 then m_n(2,q) ≤ (n-1)q+ n-(q+1)/2

Table 3. [6,7] Theorem

q / n	**41**
2	42
3	46 _ 83
4	73 _ 124
5	104 _ 165
6	136 _ 206
7	168 _ 247
8	204 _ 288
9	237 _ 329
10	270 _ 370
11	308 _ 411
12	341 _ 452
13	382 _ 493
14	417 _ 534
15	455 _ 575
16	491 _ 616
17	529 _ 657
18	569 _ 698
19	607 _ 739
20	644 _ 780
21	682 _ 821
22	727 _ 862
23	768 _ 904
24	805 _ 946
25	847 _ 988
26	889 _ 1030
27	928 _ 1072
28	974 _ 1114
29	1018 _ 1156
30	1066 _ 1198
31	1105 _ 1240
32	1151 _ 1282
33	1193 _ 1324
34	1237 _ 1366
35	1286 _ 1408
36	1330 _ 1450
37	1379 _ 1492
38	1431 _ 1534
39	1481 _ 1576
40	1535 _ 1618

5.2 [2] Definition

(i) A linear code is a subspace of V(k, q).
(ii) If dim C= n then C is a [k, n]-code or $[k, n]_q$-code or if d(C) = d, it is a [k, n, d]-code or $[k, n, d]_q$-code.

5.3 [2,9] Theorem

There exists a projective $[k, n ,d]_q$ -code if and only if there exists an(k, k - d)-arc in PG(n - 1,q)

6 New results

6.1 [3,9] Code of dimension n=3 over GF41

[42,3,39] _code , [46,3,43] _code , [73,3,70] _code , [104,3,101] _code , [136,3,133] _code , [168,3,165] _code , [204,3,201] _code , [237,3,234] _code , [270,3,267] _code , [308,3,305] _code , [341,3,338] _code , [382,3,379] _code , [417,3,414] _code , [455,3,452] _code , [491,3,488] _code , [529,3,526] _code , [569,3,566] _code , [607,3,604] _code , [644,3,641] _code , [682,3,679] _code , [727,3,724] _code , [768,3,765] _code , [805,3,802] _code , [847,3,844] _code , [889,3,886] _code , [928,3,925] _code , [974,3,971] _code , [1018,3,1015] _code , [1066,3,1063] _code , [1105,3,1102] _code , [1151,3,1148] _code , [1193,3,1190] _code , [1237,3,1234] _code , [1286,3,1283] _code , [1330,3,1327] _code , [1379,3,1376] _code , [1431,3,1428] _code , [1481,3,1478] _code , [1535,3,1532] _code , [1597,3,1594] _code , [1723,3,1720] _code

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