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Adomian decomposition method for solving nonlinear fractional delay differential equations

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Abstract: In this survey, presented numerical method said Adomian decomposition method (ADM) for settling nonlinear fractional delay differential equations (NFDDEs) .In this status use the fractional derivative with the Riemann-Liouville fractional (RLF) derivative and submitted the solution of some applications by numerically method. These results appear that the suggestion technique is very effective and simple to implement.

1 Introduction

In the prior time displayed the fractional calculus (FC) of (integrals and derivative in real or complex) in many fields of science and engineering. The differential and integral equation (DE and IE) have been provided, other types problems in sense special functions of mathematical physics [1]. Moreover, utilized the delay differential equations (DDEs) and fractional delay differential equations (FDDEs) in other fields of many papers [2-8]. Also like informed numerical method (ADM) [9], introduce an effective techniques for locating explicit and numerical solutions of a wider and general class of differential systems representing real physical problems [10,11], several methods have been forward [12-15] .In proceeding, take active technique for solving (NFDDE) by (ADM) through introduce derived in convergent series to show the efficient technique of (ADM) by solving some applications of (NFDDEs).

2 Main facts of (FC)

In the following reviewed basic definition and properties of fractional integral and derivatives [16].

Definition (1): The (RLF) integral operator of order $\alpha > 0$:

$$I_t^{\alpha}[Ƕ(t)] = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} Ƕ(x)\, dx \ \ , x > 0 \, , \alpha > 0 \quad (1)$$

$$I_t^{0}[Ƕ(t)] = Ƕ(t) \quad (2)$$

Definition (2): The (RLF) derivative operator of order > 0 :

$$D^{\alpha}\,[Ƕ(t)] = \frac{1}{\Gamma[n-\alpha]} \frac{d^n}{dt^n} \int_0^t ((t-x)^{n-\alpha-1} Ƕ(x)\, dx \quad , n \in Z \ , n-1 < \alpha \leq n \quad (3)$$

Definition (3): The Caputo fractional derivative (CFD) operator of order α as:

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$$^{c}D_t^{\alpha}\,[\text{Ƕ}(t)] = \frac{1}{\Gamma[n-\alpha]}\int_0^t ((t-x)^{n-\alpha-1}\,\frac{d^n}{dt}\text{Ƕ}(t)\,dt \quad , n\in Z\;, n-1<\alpha\le n \tag{4}$$

(CFD) has auxiliary special:

$$I^{\alpha}[\;^{c}D_t^{\alpha}[\text{Ƕ}(t)] = \text{Ƕ}(t) - \sum_{k=0}^{\infty}\text{Ƕ}^{(k)}\;(0^+)\frac{t^k}{k!}\;, n\in z, n-1<\alpha\le n \tag{5}$$

(CFD) operator is a linear operation when $\in Z$:

$$^{c}D_t^{\alpha}\,[\gamma\text{Ƕ}(t) + \delta\text{Ƿ}(t)] = \gamma\;^{c}D_t^{\alpha}\,\text{Ƕ}(t) + \;\delta\;^{c}D_t^{\alpha}\text{Ƿ}(t) \tag{6}$$

Where γ, δ are constants, the Caputo's derivative

$$^{c}D_t^{\alpha}\,K = 0\;,\ \text{K constant} \tag{7}$$

$$^{c}D_t^{\alpha}[t^n] = \begin{cases}\frac{\Gamma[1+n]}{\Gamma[1+n-\alpha]}\,t^{n-\alpha} & ,\quad n\in N\;, n\ge[\alpha]\\ 0 & ,\quad n\in N_0\;, n\ge[\alpha]\end{cases} \tag{8}$$

$[\alpha]\ge\alpha$, $[\alpha]$ smallest , $N = \{0,1,2,3,\ldots.\}$.

3 Prime facts of (ADM)

The main idea of the (ADM) by informed as:

$$L\,y + R\,y + N\,y = W\,y\;\rightarrow\;L\,y + R\,y + N\,y = G\;,\;W\,y = G \tag{9}$$

L: The highest-order derivative which has L^{-1}.
R: A linear differential operator ,nonhomogeneous term.
N: A nonlinear operator.
W: The general nonlinear ordinary differential operator.
G: afford objective.
Taking L^{-1} of second term of (3.1) find:

$$y = \beta + L^{-1}\;G - L^{-1}\;R\,y - \;L^{-1}\,N\,y \tag{10}$$

β : The settle $L\,y = 0$,with initial-boundary conditions.
Now , the major idea of matter decomposition of $N\,(y)$ by (ADM) define:

$$y = \sum_{n=0}^{\infty}\mu^n\,y_n \quad , \mu\,, y_0\,, y_{1,\ldots}\;,\;N\,(y) = \sum_{n=0}^{\infty}\mu^n\,A_n \tag{11}$$

Where $N\,(y)$ expanding by Maclurian series according to μ and A_n obtain as:

$$A_n = \frac{1}{n!}\frac{d^n}{d\mu^n}\;[N\,(\sum_{n=0}^{\infty}\mu^n\,y_n\;\;)]_{\mu=0} \tag{12}$$

A_n: said Adomian polynomials used to find all nonlinearity terms.
$N\,(y) = \;w(y)$ then the Adomian polynomials offered as:

$$\begin{aligned}A_0 &= w(y_0)\\ A_1 &= y_1 w'(y_0)\\ A_2 &= y_2 w'(y_0) + \frac{y_1^2}{2!}\;w''(y_0)\\ A_3 &= y_3 w'(y_0) + y_1 y_2 w''(y_0) + \frac{y_1^3}{3!}\;w'''(y_0)\end{aligned} \tag{13}$$

Yet, can rewrite (3.2) as:

$$y = \beta + L^{-1}\;G - \mu\,L^{-1}\;R\,y - \mu\,L^{-1}\,N\,y \tag{14}$$

$$\sum_{n=0}^{\infty}\mu^n\,y_n \;\; = \;\beta + L^{-1}\;G - \mu\,L^{-1}\;R\;\sum_{n=0}^{\infty}\mu^n\,y_n\;\; - \mu\,L^{-1}\;\sum_{n=0}^{\infty}\mu^n\,A_n \tag{15}$$

The coefficients of equal powers for μ are Equating, find:

$$\begin{aligned}y_0 &= \;\beta + L^{-1}\;G\\ y_1 &= -\,L^{-1}\;R(\,y_0) - \;L^{-1}\,(A_0)\\ y_2 &= -\,L^{-1}\;R(\,y_1) - \;L^{-1}\,(A_1)\\ y_n &= -\,L^{-1}\;R(\,y_{n-1}) - \;L^{-1}\,(A_{n-1}), n\ge 1\end{aligned} \tag{16}$$

At the last, the N-term expressed the approximate solution as:

$$\mathcal{Q}_N\,(T) = \;\sum_{n=0}^{N-1}y_n(T)\;, N\ge 1 \tag{17}$$

The exact solution introduced by:

$$y(t) = \lim_{N\to\infty}\mathcal{Q}_N\,(T) \tag{18}$$

4 Methodology

In this part, offered the approximate solution of (FDDEs):

$D_t^{\alpha}\, y(t) = N\,(t, y(t), y(Q(t))$

$$y(t) = \zeta(t) \quad ,\tau \le t \le 0 \tag{19}$$

$$y^{i}(0) = y_0^{i} \quad , i = 0,1,\dots,n-1\,,\ n-1 < \alpha \le n \tag{20}$$

Taking I_t^{α} of both sides (19) get:

$$y(t) = I_t^{\alpha} N\,(t, y(t), y\big(Q(t)\big) + \sum_{k=0}^{n-1} y(0^{+})\frac{t^k}{k!} \tag{21}$$

Yet, proffered the solution by (ADM) as:

$$y\,(t) = \sum_{n=0}^{\infty} y_n(t) \tag{22}$$

And the nonlinear part expressed by Adomian polynomials as:

$$N\,(t, y(t), y\big(Q(t)\big) = \sum_{n=0}^{\infty} A_n \tag{23}$$

Where

$$A_n = \frac{1}{n!}\frac{d^n}{d\mu^n}\left[N\left(t, \sum_{n=0}^{\infty}\mu^n\, y_n \quad , \sum_{n=0}^{\infty}\mu^n\, y_n\big(Q(t)\big)\quad\right)\right]_{\mu=0} \tag{24}$$

Now, substitute (22) in (23) find :

$$\sum_{n=0}^{\infty} y_n(t) \ = \sum_{k=0}^{n-1} y(0^{+})\frac{t^k}{k!} + I_t^{\alpha}\big(\sum_{n=0}^{\infty} A_n\big) \tag{25}$$

Lastly, this techniques getted from (ADM) the recursive connection as:

$$y_0(t) = \zeta(t) + \sum_{k=0}^{n-1} y(0^{+})\frac{t^k}{k!} + I_t^{\alpha} w(t) \tag{26}$$

$$y_{n+1}(t) = I_t^{\alpha}\, A_n \quad ,n \ge 0 \tag{27}$$

4.1 Algorithm

1. Applied (ADM) by using (12)
2. Determined the nonlinear part A_n of (NFDDEs).
3. Find y (t) by employ (27).
4. After solving step (3) obtain (22).

5 Numerical applications

In the following part, solve test some application for (NFDDEs) of (ADM) by use algorithm.

Application (5.1): Let the 1st order of (NFDDE) like:

$$D_t^{\alpha}\, y(t) = 1 + 2\, y^2\left(\frac{t}{2}\right),\ 0 \le t \le 1\,, 0 < \alpha \le 1 \tag{28}$$

$$y(0) = 0 \tag{29}$$

Solution: solve (28) by (ADM) **use** algorithm like:

$$N(y) = 2\, y^2\left(\frac{t}{2}\right) = 2A_0 \tag{30}$$

$$y_{n+1}(t) = 2\, I_t^{\alpha} A_n \tag{31}$$

After some steps find (22) as:

$$y_0(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$y_1(t) = \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \tag{32}$$

$$y_2(t) = \frac{t^{\alpha 5}}{\Gamma(5\alpha+1)}$$

$$\to y(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{\alpha 5}}{\Gamma(5\alpha+1)} + \cdots \tag{33}$$

When $\alpha = 1$ by using Taylor series get the exact solution $y(t) = sinht$

$$:\to y(t) = t + \frac{t^3}{6} + \frac{t^5}{120} + \cdots \tag{34}$$

Lastly get the approximate solution $\alpha = 0.5,0,75,1$ and comparison with exact solution show that by Figure 1.

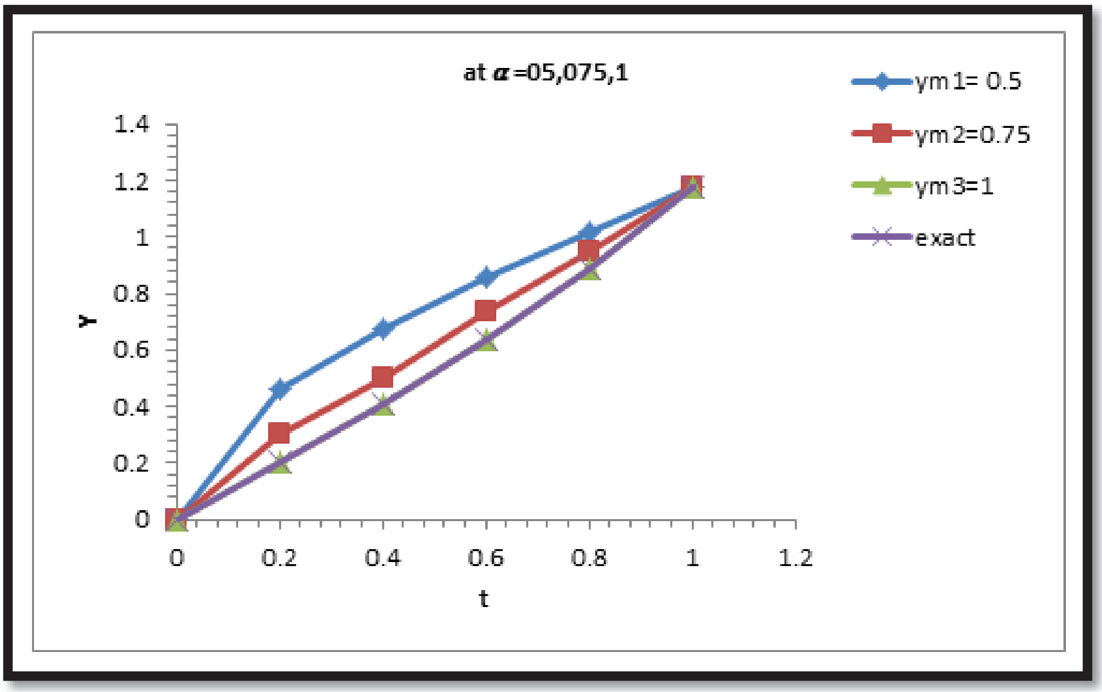

Fig. 1. 2D plots graphs of the approximate solution when α =05,075,1 and exact solution $\boldsymbol{y(t) = sinht}$.

Application (5.2): Take the 3[th]order of (NFDDE) as:

$$D_t^{3\alpha}\, y(t) = -1 + 2\, y^2\left(\frac{t}{2}\right),\ \ 0 \le t \le 1\,, 0 < \alpha \le 1 \tag{35}$$

$$y(0) = 0\ , \frac{dy(0)}{dt} = 1\,, \frac{d^2y(0)}{dt^2} = 0 \tag{36}$$

Solution: solve (35) by (ADM) **use** algorithm obtain:

$$y_0(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \tag{37}$$

$$y_{n+1}(t) = 2\, I_t^{3\alpha} A_n \tag{38}$$

After some steps find (22) as:

$$\rightarrow y(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} - \frac{t^{7\alpha}}{\Gamma(7\alpha+1)} + \cdots \tag{39}$$

When $\alpha = 1$ by using Taylor series get the exact solution $y(t) = sint$

$$\rightarrow y(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \cdots \tag{40}$$

Lastly get the approximate solution $\alpha = 0.5,0,75,1$ and comparison with exact solution performed by Figure 2.

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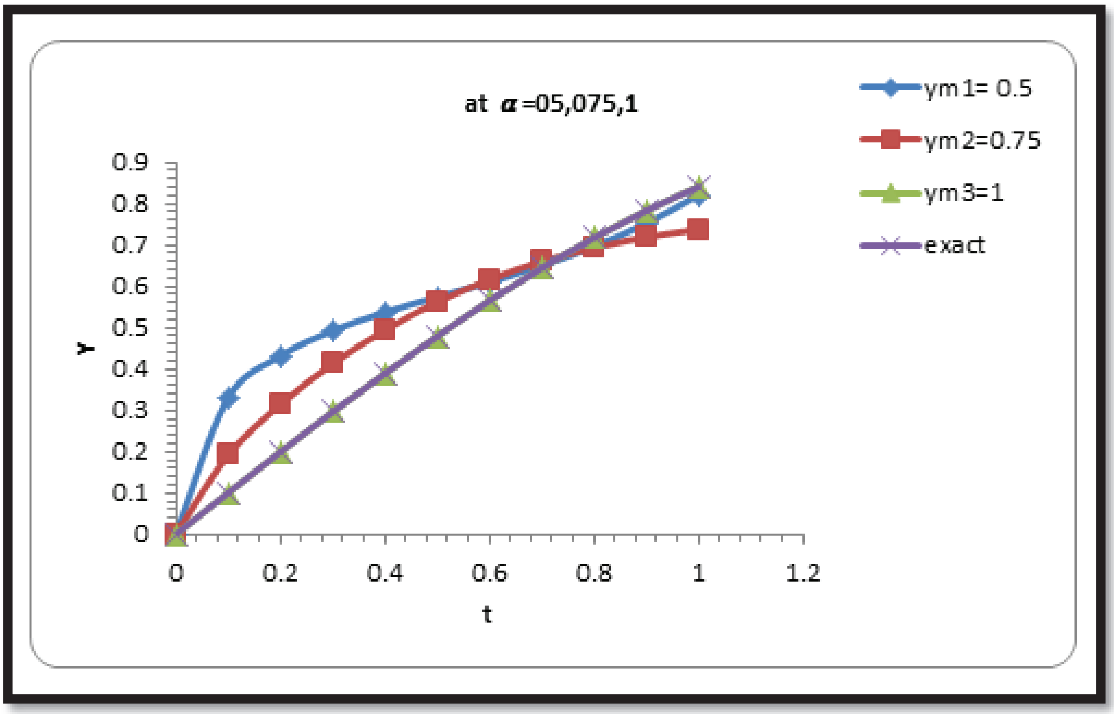

Fig. 2. 2D plots graphs of the approximate solution when α =05, 075, and 1. And exact solution $\boldsymbol{y(t) = sint}$

Application (5.3): Let the 1st order of (NFDDE) like:

$$D_t^{\alpha}\, y(t) = 1 - 2\, y^2\left(\frac{t}{2}\right),\ \ 0 \le t \le 1\,, 0 < \alpha \le 1 \tag{41}$$

$$y(0) = 0\ ,\frac{dy(0)}{dt} = 0 \tag{42}$$

Solution: solve (41) by (ADM) **use** algorithm find:

$$N(y) = -2\, y^2\left(\frac{t}{2}\right) = -2A_0 \tag{43}$$

$$y_{n+1}(t) = -2\, I_t^{\alpha} A_n \tag{44}$$

After some steps find (22) as:

$$y_0(t) = 1 + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{45}$$

$$y_1(t) = -t^{2\alpha} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - \frac{t^{6\alpha}}{\Gamma(6\alpha+1)} \tag{46}$$

$$\rightarrow y(t) = 1 - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - \frac{t^{65}}{\Gamma(6\alpha+1)} + \cdots \tag{47}$$

When $\alpha = 1$ by using Taylor series get the exact solution $y(t) = cost$

$$\rightarrow y(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots \tag{48}$$

Lastly get the approximate solution $\alpha = 0.5,0,75,2$ and comparison with exact solution viewed by Figure 3.

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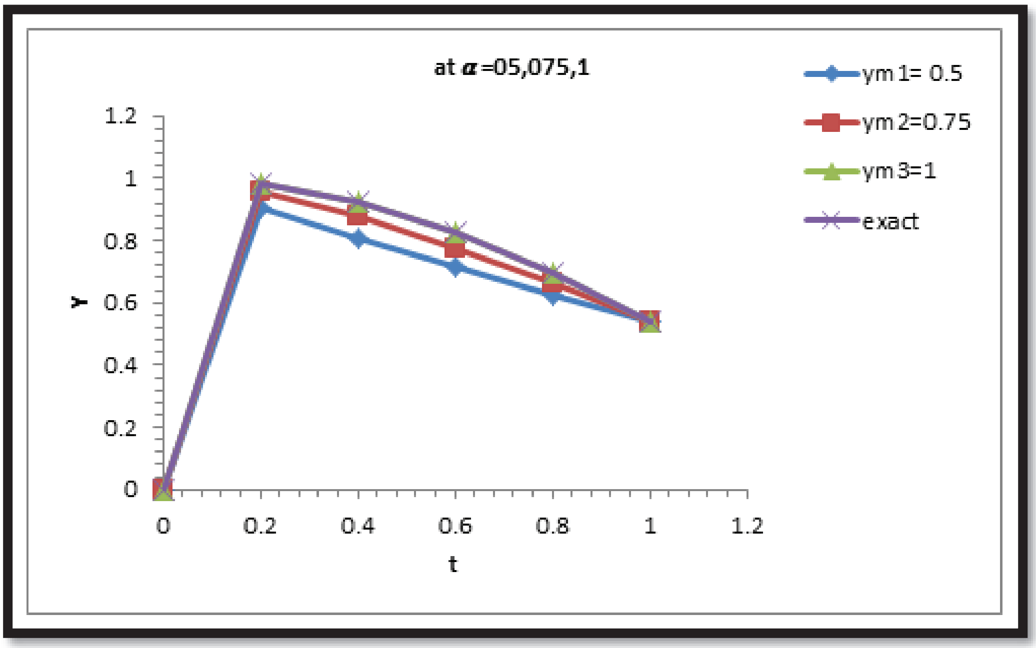

Fig. 3. 2D plots graphs of the approximate solution when α =05,075,1 . and exact solution $\boldsymbol{y(t)} = \boldsymbol{cost}$

6 Conclusion

This work objective implied attractive algorithm for solution (NFDDEs). All numerical results obtained by a powerful technique is (ADM) saw that from compared the results when α =05,075,1 with exact solution through discussed some application with figures to understand technique efficiency.

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