Adomian decomposition method for solving nonlinear fractional delay differential equations

Ali Abdulsada Gatea1*

1Ministry of Education, Third Rusafa District, Baghdad, Iraq.

Abstract: In this survey, presented numerical method said Adomian decomposition method (ADM) for settling nonlinear fractional delay differential equations (NFDDEs). In this status use the fractional derivative with the Riemann-Liouville fractional (RLF) derivative and submitted the solution of some applications by numerically method. These results appear that the suggestion technique is very effective and simple to implement.

1 Introduction

In the prior time displayed the fractional calculus (FC) of (integrals and derivative in real or complex) in many fields of science and engineering. The differential and integral equation (DE and IE) have been provided, other types problems in sense special functions of mathematical physics [1]. Moreover, utilized the delay differential equations (DDEs) and fractional delay differential equations (FDDEs) in other fields of many papers [2-8]. Also like informed numerical method (ADM) [9], introduce an effective techniques for locating explicit and numerical solutions of a wider and general class of differential systems representing real physical problems [10,11], several methods have been forward [12-15]. In proceeding, take active technique for solving (NFDDE) by (ADM) through introduce derived in convergent series to show the efficient technique of (ADM) by solving some applications of (NFDDEs).

2 Main facts of (FC)

In the following reviewed basic definition and properties of fractional integral and derivatives [16].

Definition (1): The (RLF) integral operator of order \( \alpha > 0 \):

\[
I_1^\alpha [H_b(t)] = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} H_b(x) \, dx \quad x > 0, \alpha > 0
\]

Definition (2): The (RLF) derivative operator of order \( \alpha > 0 \):

\[
D_1^\alpha [H_b(t)] = H_b(t)
\]

Definition (3): The Caputo fractional derivative (CFD) operator of order \( \alpha \) as:

* Corresponding author: ali.iq8200@gmail.com
The main idea of the (ADM) by informed as:

\[ cD_t^\alpha [Hb(t)] = \frac{1}{\Gamma[n-\alpha]} \int_0^t ((t-x)^{n-\alpha-1} \frac{d^n}{dx^n} Hb(t)) dx, n \in Z, n-1 < \alpha \leq n \]  

(CFD) has auxiliary special:

\[ I^\alpha [ cD_t^\alpha [Hb(t)]] = Hb(t) - \sum_{k=0}^{\infty} Hb^{(k)}(0^+) \frac{t^k}{k!}, n \in Z, n-1 < \alpha \leq n \]

Now, the major idea of matter decomposition of (CFD) has auxiliary special:

\[ cD_t^\alpha \eta (t) + cD_t^\delta \phi (t) = \gamma cD_t^\alpha Hb(t) + \delta cD_t^\alpha P(t) \]

Where \( \gamma, \delta \) are constants, the Caputo’s derivative

\[ cD_t^\alpha K = 0, K \text{ constant} \]

\[ cD_t^\alpha [t^n] = \begin{cases} \frac{\Gamma[1+n]}{\Gamma[1+n-\alpha]} t^{n-\alpha}, & n \in N, n \geq [\alpha] \\ 0, & n \in N_0, n \geq [\alpha] \end{cases} \]

[\( \alpha \geq \alpha \), \( \alpha \) smallest, \( N = \{0,1,2,3,\ldots\} \).]

3 Prime facts of (ADM)

The main idea of the (ADM) by informed as:

\[ L y + R y + N y = W y \rightarrow L y + R y + N y = G, W y = G \]

L: The highest-order derivative which has \( L^{-1} \).
R: A linear differential operator, nonhomogeneous term.
N: A nonlinear operator.
W: The general nonlinear ordinary differential operator.
G: afford objective.

Taking \( L^{-1} \) of second term of (3.1) find:

\[ y = \beta + L^{-1} G - L^{-1} R y - L^{-1} N y \]

\( \beta \): The settle \( L y = 0 \), with initial-boundary conditions.

Now, the major idea of matter decomposition of \( N(y) \) by (ADM) define:

\[ y = \sum_{n=0}^{\infty} \mu^n y_n, \mu, y_0, y_1, \ldots, N(y) = \sum_{n=0}^{\infty} \mu^n A_n \]

Where \( N(y) \) expanding by Maclurian series according to \( \mu \) and \( A_n \) obtain as:

\[ A_n = \frac{1}{n!} \frac{d^n}{d\mu^n} [N (\sum_{n=0}^{\infty} \mu^n y_n)]_{\mu=0} \]

\( A_n \): said Adomian polynomials used to find all nonlinearity terms.

\( N(y) = w(y) \) then the Adomian polynomials offered as:

\[ A_0 = w(y_0) \]
\[ A_1 = y_1 w'(y_0) \]
\[ A_2 = y_2 w'(y_0) + \frac{y_1^2}{2} w''(y_0) \]
\[ A_3 = y_3 w'(y_0) + y_1 y_2 w''(y_0) + \frac{y_1^3}{3!} w'''(y_0) \]

Yet, can rewrite (3.2) as:

\[ y = \beta + L^{-1} G - \mu L^{-1} R y - \mu L^{-1} N y \]
\[ \sum_{n=0}^{\infty} \mu^n y_n = \beta + L^{-1} G - \mu L^{-1} R \sum_{n=0}^{\infty} \mu^n y_n - \mu L^{-1} \sum_{n=0}^{\infty} \mu^n A_n \]

The coefficients of equal powers for \( \mu \) are Equating, find:

\[ y_0 = \beta + L^{-1} G \]
\[ y_1 = -L^{-1} R(y_0) - L^{-1} (A_0) \]
\[ y_2 = -L^{-1} R(y_1) - L^{-1} (A_1) \]
\[ y_n = -L^{-1} R(y_{n-1}) - L^{-1} (A_{n-1}), n \geq 1 \]

At the last, the N-term expressed the approximate solution as:

\[ Q_N (T) = \sum_{n=0}^{N-1} y_n (T), N \geq 1 \]

The exact solution introduced by:

\[ y(t) = \lim_{N \to \infty} Q_N (T) \]
4 Methodology

In this part, the approximate solution of (FDDEs):

\[ D_t^\alpha y(t) = N(t, y(t), y(Q(t))) \]

\[ y(t) = \zeta(t), \quad \tau \leq t \leq 0 \]

\[ y^i(0) = y^i_0, \quad i = 0, 1, ..., n - 1, \quad n - 1 < \alpha \leq n \]

Taking \( I_t^\alpha \) of both sides (19) get:

\[ y(t) = I_t^\alpha N(t, y(t), y(Q(t))) + \sum_{k=0}^{n-1} \frac{y(0^+)^k}{k!} \]

Yet, the solution by (ADM) as:

\[ y(t) = \sum_{n=0}^{\infty} Y_n(t) \]

And the nonlinear part expressed by Adomian polynomials as:

\[ N(t, y(t), y(Q(t))) = \sum_{n=0}^{\infty} A_n \]

Where

\[ A_n = \frac{1}{n!} \frac{d^n}{d\mu^n} \left[ N(t, \sum_{n=0}^{\infty} \mu^n y_n, \sum_{n=0}^{\infty} \mu^n y_n(Q(t)) \right] \bigg|_{\mu=0} \]

Now, substitute (22) in (23) find:

\[ \sum_{n=0}^{\infty} y_n(t) = \sum_{k=0}^{n-1} \frac{y(0^+)^k}{k!} + I_t^\alpha \left( \sum_{n=0}^{\infty} A_n \right) \]

Lastly, this techniques got from (ADM) the recursive connection as:

\[ y_0(t) = \zeta(t) + \sum_{k=0}^{n-1} \frac{y(0^+)^k}{k!} + I_t^\alpha w(t) \]

\[ y_{n+1}(t) = I_t^\alpha A_n, \quad n \geq 0 \]

4.1 Algorithm

1. Applied (ADM) by using (12)
2. Determined the nonlinear part \( A_n \) of (NFDDEs).
3. Find \( y(t) \) by employ (27).
4. After solving step (3) obtain (22).

5 Numerical applications

In the following part, solve test some application for (NFDDEs) of (ADM) by use algorithm.

**Application (5.1):** Let the 1st order of (NFDDE) like:

\[ D_t^\alpha y(t) = 1 + 2 y^2 \left( \frac{t}{2} \right), \quad 0 \leq t \leq 1, \quad 0 < \alpha \leq 1 \]

\[ y(0) = 0 \]

**Solution:** solve (28) by (ADM) use algorithm like:

\[ N(y) = 2 y^2 \left( \frac{t}{2} \right) = 2A_0 \]

\[ y_{n+1}(t) = 2 I_t^\alpha A_n \]

After some steps find (22) as:

\[ y_0(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} \]

\[ y_1(t) = \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \]

\[ y_2(t) = \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} \]

\[ \rightarrow y(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} + \cdots \]

When \( \alpha = 1 \) by using Taylor series get the exact solution \( y(t) = \sin ht \)

\[ : \rightarrow y(t) = t + \frac{t^3}{6} + \frac{t^5}{120} + \cdots \]
Lastly get the approximate solution $\alpha = 0.5, 0.75, 1$ and comparison with exact solution show that by Figure 1.

![Graph showing approximate and exact solutions](image)

**Fig. 1.** 2D plots graphs of the approximate solution when $\alpha = 0.5, 0.75, 1$ and exact solution $y(t) = \sinht$.

**Application (5.2):** Take the 3rd order of (NFDDE) as:

$$D_t^{2\alpha} y(t) = -1 + 2 y^2 \left(\frac{t}{2}\right), \ 0 \leq t \leq 1, \ 0 < \alpha \leq 1 \quad (35)$$
$$y(0) = 0, \ \frac{dy(0)}{dt} = 1, \ \frac{d^2 y(0)}{dt^2} = 0 \quad (36)$$

**Solution:** solve (35) by (ADM) use algorithm obtain:

$$y_0(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \quad (37)$$
$$y_{n+1}(t) = 2 l_t^{3\alpha} A_n \quad (38)$$

After some steps find (22) as:

$$y(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} - \frac{t^{7\alpha}}{\Gamma(7\alpha+1)} + \cdots \quad (39)$$

When $\alpha = 1$ by using Taylor series get the exact solution $y(t) = \sin t$

$$y(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \cdots \quad (40)$$

Lastly get the approximate solution $\alpha = 0.5, 0.75, 1$ and comparison with exact solution performed by Figure 2.
Application (5.3): Let the 1st order of (NFDDE) like:

\[ D_t^\alpha y(t) = 1 - 2y^2 \left( \frac{t}{2} \right), \quad 0 \leq t \leq 1, \quad 0 < \alpha \leq 1 \]  

(41)

\[ y(0) = 0, \quad \frac{dy(0)}{dt} = 0 \]  

(42)

Solution: solve (41) by (ADM) use algorithm find:

\[ N(y) = -2y^2 \left( \frac{t}{2} \right) = -2A_0 \]  

(43)

\[ y_{n+1}(t) = -2t^n A_n \]  

(44)

After some steps find (22) as:

\[ y_0(t) = 1 + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \]  

(45)

\[ y_1(t) = -t^{2\alpha} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - \frac{t^{6\alpha}}{\Gamma(6\alpha+1)} \]  

(46)

\[ \rightarrow y(t) = 1 - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - \frac{t^{6\alpha}}{\Gamma(6\alpha+1)} + \cdots \]  

(47)

When \( \alpha = 1 \) by using Taylor series get the exact solution \( y(t) = \cos t \)

\[ \rightarrow y(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots \]  

(48)

Lastly get the approximate solution \( \alpha = 0.5, 0.75, 2 \) and comparison with exact solution viewed by Figure 3.
Fig. 3. 2D plots graphs of the approximate solution when $\alpha = 0.075, 1$ and exact solution $y(t) =$ cost

6 Conclusion

This work objective implied attractive algorithm for solution (NFDDEs). All numerical results obtained by a powerful technique is (ADM) saw that from compared the results when $\alpha = 0.075, 1$ with exact solution through discussed some application with figures to understand technique efficiency.

References


