BIO Web of Conferences **97**, 00165 (2024) https://doi.org/10.1051/bioconf/20249700165
ISCKU 2024

tg-Radical Supplemented Modules

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Abstract. In this work, we define tg-radical supplemented modules and cofinitely tg-radical supplemented modules. We investigate some properties of these modules. In addition, we present examples separating the tg-radical supplemented modules, g-supplemented modules, and ⨁-g-Rad-supplemented modules and also show the equality of these modules for projective and finitely generated modules. We give a characterization of cofinitely tg-radical supplemented modules. Furthermore, for any ring R, we show that any finite direct sum of tg-radical supplemented R-modules is tg-radical supplemented and that any direct sum of cofinitely tg-radical supplemented R-modules is a cofinitely tg-radical supplemented module.

1. Introduction

Throughout the paper, all rings are associative rings with identity and all modules are unital left modules. Let R be a ring and let M be an R-module. We indicate a submodule N of M by $N \leq M$ besides a proper submodul W of M by $W < M$. Presume M is an R-module besides presume $N \leq M$. A submodule $L \leq M$ is said to be essential in M, symbolized by $L \trianglelefteq M$, if $L \cap N \neq 0$ for all non-zer0 submodul $N \leq M$. As a dual to the concept of an essential submodul, a subm0dule N of M is named small in M, symbolized by $N \ll M$, if $M \neq N + K$ for any proper submodule K 0f M see [1]. Following [3], a submodul N of M is named a generalized small (briefly, g-small) submodule 0f M if for every $T \trianglelefteq M$ with $M = N + T$ implies that $T = M$, this is written by $N \ll_{\mathrm{g}} M$ (in [2] it is named an e-small subm0dule of M besides indicated as $N \ll_{\mathrm{e}} M$). If T is essential and maximal subm0dule 0f M then T is said to be a generalized maximal subm0dule 0f M. The intersection 0f all generalized maximal submoduls 0f M is named the generalized radical 0f M besides indicated by $Rad_{\mathrm{g}}M$ [3] (In [2] it is indicated using $Rad_{\mathrm{e}}M$). If M has no essential maximal submoduls, at that time we indicate $Rad_{\mathrm{g}}M = M$.

Presume that M be an R-module besides let $N \leq M$. If there is a submodul G of M such that $M = N + G$ besides $N \cap G = 0$, then N is said to be direct summand of M besides indicated be $M = N \oplus G$. F0r any modul M, one has $M = M \oplus 0$. Using $Rad(M)$ one symbolize the radical 0f M. An R-modul M is called a simple if M has no pr0per submoduls with distinct zer0. Suppose that M be an R-modul. One say that M is a (semi) hollow modul if every (finitely generated) pr0per submodul 0f M is small in M. Besides, M is said to be l0cal module if M has the largest submodul, i.e., the proper submodul that c0ntains all 0ther pr0per submoduls. A modul M is said to be distributive [11] if, f0r any submoduls W, G, and N of M, we have $N + (W \cap G) = (N + W) \cap (N + G)$. Or, equivalently, $N \cap (W + G) = (N \cap W) + (N \cap G)$. Let M be an R-modul besides let X besides Y be submoduls of M. If $M = X + Y$ and Y is minimal with respect t0 this pr0perty 0r, equivalently. $M = X + Y$ and $X \cap Y \ll Y$, then Y is said to be a supplement of X in M [1]. A modul M is said to be a supplemented m0dule if every submodule 0f M has a supplement (see f0r example [1, 4, 5]). A submodul Y is said to be a generalized supplement (Rad-supplement) of X in M if $M = X + Y$ besides $X \cap Y \leq Rad(Y)$ [12].

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A modul M is said to be generalized supplementd or simply a Rad-supplementd modul if every submodul of M has a generalized supplement (Rad-supplement) besides it is obvious that any supplementd modul is a generalized supplementd. In [3], a modul M is named g-supplemented if each submodul U 0f M has a g-supplement V in M, that is, $M = U + V$ and $U \cap V \ll_g V$. Let M be an R-modul besides $U, V \leq M$. If $M = U + V$ and $U \cap V \leq Rad_g V$, then V is named a g-radical supplement of U in M. If every submodule 0f M has a g-radical supplement in M, then M is named a g-radical supplemented module [9]. A modul M is said to be ⊕-supplemented if every submodul 0f M has a supplement which is a direct summand of M. M is said to be ⊕-g-supplemented if every subm0dule 0f M has a g-supplemented which is a direct summand of M [14]. A modul M is said t0 be a ⊕-g-Rad-supplemented module if every submodul 0f M has a g-radical supplement in the f0rm 0f a direct summand in M [14]. In the current paper, we generalize these module. A submodul N 0f an R-modul M is said to be cofinite if M/N is finitely generated. A modul M is said to be cofinitely g-supplemented [13] if every cofinite submodul of M has a g-supplement in M.

In the next secti0n, we define tg-radical supplemented m0dules and examine the relati0nship among these moduls, g-supplementd moduls, and ⊕-g-Rad-supplemented moduls. Every ring R, we sh0w that any finite direct sum 0f tg-radical supplementd moduls is a tg-radical supplementd module besides establish conditions for tg-radical supplementd modules specifying the fact0r moduls of these tg-radical supplementd moduls.

In the latest section, we define cofinitely tg-radical supplementd moduls besides examine their relati0nship with cofinitely g-supplementd modules. We too illustration that any direct sum 0f c0finitely tg-radical supplementd R-moduls is too a cofinitely tg-radical supplementd R-modul f0r any ring R.

We state the next lemma which is contained in [2, Proposition 2.3].

Lemma 1.1. Let N be a subm0dule 0f a m0dule M. The foll0wing are equivalent.

(1) $N \ll_g M$,

(2) if $M = X + N$, then X is a direct summand 0f M with $\frac{M}{X}$ a semisimple m0dule.

It is obvious that Lemma 1.1(2) is equal to if $M = X + N$, at that time $M = X \oplus Y$ f0r s0me semisimple submodule Y of M.

Evidently, M is projective semisimpl m0dule if and 0nly if $M \ll_g M$. A ring R is l0cal if ${}_RR$ (0r R_R) is a l0cal m0dule.

The foll0wing lemma follows from [2, Proposition 2.5] besides from [9].

Lemma 1.2. Supposing M is a modul. Now

(1) F0r subm0dules N, K, L 0f M with $K \leq N$, we have
 (a) If $N \ll_g M$, then $K \ll_g M$ besides $N/K \ll_g M/K$.
 (b) $N + L \ll_g M$ if and 0nly if $N \ll_g M$ and $L \ll_g M$.
(2) If $K \ll_g M$ besides $f: M \to N$ is a homom0rphism, then $f(K) \ll_g N$. In particular, if $K \ll_g M \leq N$, then $K \ll_g N$.
(3) Let N, K, L, besides T be submoduls of M. If $K \ll_g L$ besides $N \ll_g T$, then $K + N \ll_g L + T$.
(4) Let $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$ and $M = M_1 \oplus M_2$. At that time $K_1 \oplus K_2 \ll_g M_1 \oplus M_2$ if and **0**nly if $K_1 \ll_g M_1$ besides $K_2 \ll_g M_2$.

Definition 1.3. ([2], [9]) Presume M is a m0dul. Define

$$Rad_g(M) = \cap \{N \trianglelefteq M \mid N \text{ is maximal in } M\}.$$

If M have n0 maximal essential subm**0**dules, now we den0te $Rad_g(M) = M$.

Clearly, $Rad(M) \leq Rad_g(M)$ and $Soc(M) \leq Rad_g(M)$. For an arbitrary ring R, let $Rad_g(R) = Rad_g({}_RR)$. For more information about $Rad_g(M)$ see for example [6]. In the following we use g-small submodules to characterize $Rad_g(M)$.

Theorem 1.4. Let M be an R-modules. Then $Rad_g(M) = \sum_{N \ll_g M} N$.

Proof. See [3, Lemma 5].

Lemma 1.5. Let M besides N be modules. Now

(1) If $f: M \to N$ is an R-homomorphism, then $f(Rad_g(M)) \leq Rad_g(N)$.
(2) If every pr0per essential subm**0**dule 0f M is c0ntained in a maximal subm**0**dule 0f M, then $Rad_g(M)$ is the unique largest g-small subm**0**dule 0f M.

Proof. See [2, Corollary 2.11].

Remark 1.6. It is obvious that, in general, $Rad_g(M)$ need n0t be g-small in M. But if M is a coatomic module, i.e. every pr0per subm0dule 0f M is c0ntained in a maximal subm0dule **0**f M, then $Rad_g(M)$ is g-small in M using Lemmaa 1.5(2).

Lemma 1.7. If $M = \oplus_{i \in I} M_i$ then $Rad_g(M) = \oplus_{i \in I} Rad_g(M_i)$.

Proof. See [9, Lemma 4].

2. tg-Radical Supplemented Modules

In this section, we introduce the concept of tg-radical supplemented modules. In [8], a module M is named a t-generalized supplementd m0dule if every submodul 0f M has a generalized supplement which is too a supplement in M. Similar to [8] we give the following definition.

Definition 2.1. Let M be an R-module. The module M is called a tg-radical supplemented module if every submodule of M has a g-radical supplement which is also a g-supplement in M. Let M be a module and let $K, N \leq M$. Then K is called tg-radical supplement of N if $M = N + K$, $N \cap K \leq Rad_g K$ and K is g-supplement. The module M is said to be tg-radical supplemented module if every subm0dule 0f M has a tg-radical supplement.

Clearly, the ⊕-g-Rad-supplemented moduls are tg-radical supplementd. Though, the c0nverse implication fails t0 be true. This is sh0wn in Examplee 2.13.

It is too obvious that, despite the fact that all g-supplemented modul is tg-radical supplemented, the converse asserti0n is n0t always true. As the hollow besides l0cal moduls are g-supplemented, they are tg-radical supplementd moduls.

It is well kn0wn that every ⊕-g-supplemented m0dule is ⊕-g-Rad-supplemented. Now we designate a situati0n in which the reverse asserti0n is exact.

Lemma 2.2. Presume M is a finitely generated R-modul, at that time M is ⊕-g-Rad-supplemented if and only if M is ⊕-g-supplemented.

Proof. (⟹) Suppose that A is a submodul 0f M. As M is ⊕-g-Rad-supplementd, then there is a g-radical supplemented G of A where G is direct summand in M. Henceforth, there are submoduls G besides W 0f M with

$$M = A + G, \quad A \cap G \leq Rad_g\, G, \quad \text{and} \quad M = G \oplus W.$$

Since M is finitely generatd, one concludes G is finitely generated besides Rad $G \ll_g G$. At that time, $A \cap G \ll_g G$ besides G is a g-supplement 0f A in M. Accordingly, M is ⊕-g-supplementd.

(⟸) Obvious.

The next lemmaa is used t0 pr0ve The0rem 2.6.

Lemma 2.3. Presume M is an R-module with $M = M_1 \oplus M_2$ besides let G and W be submoduls 0f M_1 such that G is a g-supplement of W in M_1. At that time G is a supplement of $M_2 + W$ in M.

Proof. Suppose that $M_2 + W + B = M$ with $B \leq G$. Hence,

$$M_1 = M_1 \cap M = M_1 \cap (W + B + M_2) = W + B + (M_1 \cap M_2) = W + B.$$

Since $B \leq G$ and G is a g-supplement of W in M_1, we get $B = G$. Then, G is a g-supplement 0f $M_2 + W$ in M.

Lemma 2.4. Suppose $M = M_1 \oplus M_2$. If X is a g-supplement submodule in $M_1$and Y is a g-supplement submodule in M_2, then $X + Y$ is a g-supplement submodule in M.

Proof. Let X is a g-supplement 0f A in M_1 plus Y is a g-supplement 0f B in M_2. Now,

$$M_1 = A + X, \quad A \cap X \ll_g X, \quad \text{besides} \quad M_2 = B + Y, \quad B \cap Y \ll_g Y.$$

Since $M_1 = A + X$ and $M_2 = B + Y$, one has

$$M = M_1 + M_2 = A + B + X + Y.$$

It is easy t0 check thatt $(A + X + B) \cap Y \ll_g Y$ besides $(B + Y + A) \cap X \ll_g X$. Hence,

$$(A + B) \cap (X + Y) \subseteq (A + B + Y) \cap X + (A + B + X) \cap Y \ll_g X + Y.$$

Then, $X + Y$ is a g-supplementt 0f $A + B$ in M.

The foll0wing result generalizes Lemmaa 2.4 besides can easily be pr0ved.

Corollary 2.5. Suppose that $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$. F0r $1 \leq i \leq n, if\ L_i$ is a g-supplement submodule in M_i, then $L_1 + L_2 + \ldots + L_n$ is a g-supplement submodul in M.

Theorem 2.6. F0r arbitrary ring R, the finite direct sum 0f tg-radical supplemented R-modules is tg-radical supplemented.

Proof. Suppose that n be anyy p0sitive integer, let $\{Mi\}_{1 \leq i \leq n}$ bee any finit colection 0f tg-radical supplementd R-moduls, besides assume that $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$. Suppose that $n = 2$. Also let $M = M_1 \oplus M_2$ and A be any submodule of M. Thus, $M = M_1 + M_2 + A$. Since M_2 is tg-radical supplementd, one can say that $M_2 \cap (M_1 + A)$ has a g-radical supplement U in M_2 such that U is a suplement in M_2. Then, U is a g-radical supplement 0f $M_1 + A$ in M. As M_1 is tg-radical supplementd, $M_1 \cap (U + A)$ has a g-radical suplement V in M_1 such that V is a g-supplement in M_1. Thus, one c0nclude that $U + V$ is a g-radical supplement 0f A in M (see [9]). As U is a g-supplement in M_2 besides V is a g-supplement in M_1, by Lemm 2.4, $U + V$ is a g-supplement in M. Henceforth, M is tg-radical supplementd. The remaining part of the pr00f is c0mpleted by inducti0n on n.

The relati0nship between the c0ncepts of tg-radical supplementd besides g-supplementd is expressed by thee next lemm.

Lemma 2.7. Presume that M is a finitely generated module. Then M is tg-radical supplemented if and only if M is g-supplemented.

Proof. ($\Longrightarrow$) Assume that G be any submodul 0f M. As M is tg-radical supplementd, there exist $W \leq M$ such that $M = G + W, G \cap W \subseteq Rad_g\ W$, and W is a g-supplement in M. As M is finitely generated, we get $Rad_g\ M \ll M$ by [13, Lemma 14], or by Remark 1.6, since every finitely generated is coatomic. Henceforth, $G \cap W \subseteq Rad_g\ W \subseteq Rad_g M \ll_g M$ and we get $G \cap W \ll_g W$. This earnings that W is a g-supplement 0f G in M besides, as a result, M is g-supplementd.

($\Longleftarrow$) Clear fr0m the definiti0ns.

Lemma 2.8. Presume that M is an R-module. If $Rad_g M = M$, then M is tg-radical supplemented.

Proof. Suppose that A be any submodule of M. As $A + M = M$ besides $A \cap M \subseteq M = Rad_g M$, one concludes M is a g-radical supplementt 0f A. 0n the 0ther hand M is a g-supplementt 0f 0. Henceforth, M is tg-radical supplementd. □

It is easy t0 see that for any semiholl0w m0dule is tg-radical supplementd. We n0w present certain examples 0f moduls which are tg-radical supplementd but not supplementd. As a result, the next exampless are given t0 separate the structures 0f tg-radical supplementd, g-supplementd, and ⊕-g-Rad-supplementd moduls.

Example 2.9. Presume a $\mathbb{Z}$-m0dule $\mathbb{Q}$. As $\mathbb{Q}$ has no maximall submodul, one has $Rad_g\ \mathbb{Q} = \mathbb{Q}$. Using Lemma 2.8, $\mathbb{Q}$ is a tg-radical supplementd module. Though, it is well kn0wn that $\mathbb{Q}$ is not g-supplemented see [9].

Example 2.10. Suppose that M is a non-t0rsion $\mathbb{Z}$-modulee with $Rad_g\ M = M$. As $Rad_g\ M = M$, we determine that M is tg-radical supplementd. Together, M is not supplemented besides as a result g-supplemented.

Example 2.11. Presume a $\mathbb{Z}$-modulee $M = \mathbb{Q} \oplus \mathbb{Z}/p\mathbb{Z}$, f0r any primee p. In thiss casee $Rad_g\ M \neq M$. In addition, M is tg-radical supplemented. Besides, M is g-radical supplementd but not supplementd by [9, Example].

Definition 2.12. A module M is named completely ⊕-g-supplemented if each direct summand of M is ⊕-g-supplementd.

Evidently, completely ⊕-supplemented is completely ⊕-g-supplemented.

Example 2.13. Suppose that R is aa commutative local ring which is n0t aa valuation ring. Let x besides y be elements of R such that neither 0f them divides the 0ther. By taking a suitable qu0tient ring, wee can suppose thatt

$$(x) \cap (y) = 0 \text{ besides } xm = ym = 0,$$

where m iis a maximal ideal 0f R. Presume D is a free R-module with generators a_1, a_2, plus a_3, let E be the subm0dule generated by $xa_1 - ya_2$, besides let $M = D/E$. Then

$$M = \frac{Ra_1 \oplus Ra_2 \oplus Ra_3}{R(xa_1 - ya_2)} = (R\overline{a_1} + R\overline{a_2}) \oplus R\ \overline{a_3}.$$

Here, M is not ⊕-supplemented by [8, Example 2.4]. Henceforth, M is not ⊕-g-supplemented. At the same time, $D = Ra_1 \oplus Ra_2 \oplus Ra_3$ is completely ⊕-supplemented [10]. As a result, D is completely ⊕-g-supplemented.

Since D is completely ⊕-g-supplementd, D is g-supplementd. Since a fact0r m0dule 0f aa g-supplemented module is g-supplementd, one concludes that M is g-supplemented. Henceforth, M is tg-radical supplementd. Separately, as M is finitely generated plus not ⊕-g-supplemented, as a result M is not ⊕-g-Rad-supplemented by Lemmaa 2.2.

Lemma 2.14. Let $M = M_1 \oplus M_2$. Then M_2 is tg-radical supplemented if and 0nly if, f0r any submodule A/M_1 of M/M_1, there exists a g-supplement B in M such that

$$B \leq M_2, \qquad M = B + A, \qquad and \qquad A \cap B \subseteq Rad_g\, M.$$

Proof. ($\Longrightarrow$) Suppose that M_2 is tg-radical supplemented. Let $A/M_1 \leq M/M_1$. Since M_2 is tg-radical supplementd, there is a tg-radical supplement submodule B of $A \cap M_2$ such that B is a g-supplement in M_2. Hence, there exists $B' \leq M_2$ such that

$$M_2 = A \cap M_2 + B, \quad A \cap M_2 \cap B \subseteq Rad_g B, \text{ and } M_2 = B + B', \quad B \cap B' \ll_g B.$$

The equality $M = M_1 + M_2$ implies that $M = M_1 + A \cap M_2 + B = A + B$. On the 0ther hand, $A \cap B \subseteq Rad_g\, M$. As B is a g-supplement 0f B' in M_2 besides $M = M_1 \oplus M_2$, we conclude that B is g-supplementt 0f $M_1 + B'$ in M by Lemmaa 2.3. Then, B is a g-supplementt iin M.

($\Longleftarrow$) Assume that M/M_1 satisfies the properties of the hypothesis. Let $X \leq M_2$. Consider the submodule

$$(X \oplus M_1)/M_1 \leq M/M_1.$$

Using the hypothesis, there exists a g-supplementt Y in M such thatt

$$Y \leq M_2, \qquad M = (Y + X) \oplus M_1, \qquad \text{and} \qquad Y \cap (X + M_1) \subseteq Rad_g M.$$

Since $Y \cap X \leq Y \cap (X + M_1) \subseteq Rad_g M$ and $Y \cap X \leq Y$, we have

$$Y \cap X \leq Y \cap Rad_g M = Rad_g Y.$$

Hence, Y is a g-radical supplementt 0f X in M_2.

Presume thatt Y is a g-supplement 0f Z in M. In this casee,

$$M = Z + Y \qquad \text{and} \qquad Z \cap Y \ll Y.$$

Note thatt $M_2 = M_2 \cap M = M_2 \cap (Y + Z) = Y + M_2 \cap Z$.

Since $M_2 \cap Z \cap Y \leq Z \cap Y \ll Y$, it is not difficult t0 see that Y is a g-supplement 0f $M_2 \cap Z$ in M_2. □

Next the0rem can be obtained as a c0nsequence 0f Lemm 2.14.

Theorem 2.15. Let $M = M_1 \oplus M_2$ be a tg-radical supplemented module besides presume $A \cap M_2$ is a g-supplement in M f0r any g-supplement A in M with $M = A + M_2$. Then M_2 is tg-radical supplemented.

Proof. Suppose $B/M_1 \leq M/M_1$. Presume the submodul $B \cap M_2$ of M. As M is tg-radical supplementd, there existss aa g-radical supplementd A' of $B \cap M_2$ such thatt A' is a g-suplement in M, i.e., there exists a g-suplement A' in M such thatt

$$M = (B \cap M_2) + A' \text{ besides } (B \cap M_2) \cap A' \leq Rad_g\, A'.$$

Since $M = (B \cap M_2) + A'$, we get

$$M = M_2 + A'.$$

Suppose that $A = M_2 \cap A'$. Then

https://doi.org/10.1051/bioconf/20249700165

$$M_2 = (B \cap M_2) + (M_2 \cap A') = (B \cap M_2) + A .$$

It follows from $M = M_1 + M_2$ and $M_1 \leq B$ that

$$M = B + M_2 = B + (B \cap M_2) + (M_2 \cap A') = B + A .$$

Since $M = M_2 + A'$ and A' is aa g-supplement in M, we see that $A = A' \cap M_2$ is a g-supplement in M using the supposition. As a result, M_2 is tg-radical supplementd using Lemm 2.14. □

We n0w investigat certain conditi0ns guaranteeing that a fact0r modul of a (distributive) tg-radical supplementd modul is tg-radical supplementd.

Lemma 2.16. Let M be tg-radical supplemented modul and presume that $X \leq M$. If $(X + Y)/X$ is a supplemented submodule in M/X for any g-supplement submodule Y in, then M / X is tg-radical supplemented.

Proof. As M is tg-radical supplementd, f0r every submodul Q 0f M c0ntaining, there exists $P' \leq M$ such that, for some submodule P of M,

$$M = Q + P = P + P', \qquad Q \cap P \leq Rad_g P, \quad \text{and} \quad P \cap P' \ll_g P.$$

Since $M = Q + P$ and $X \leq Q$, we get

$$M/X = (Q + P)/X = Q/X + (P + X)/X$$

Note that $Q \cap P \leq Rad_g\, P$ and

$$Q/X \cap (P + X)/X = (Q \cap P + X)/X \leq \left(Rad_g\, P + X\right)/X \leq Rad_g((P + X)/X).$$

This implies that $(P + X) / X$ is a g-radical supplement of Q/X in M/X. On the 0ther hand, P is a g-supplemen in M and $(P + X) / X$ is a g-supplemen in M / X via the hypothesis. Then, $(P + X) / X$ is a g-radical supplementt 0f Q / X in M / X such thatt $(P + X) / X$ is a g-supplement in M / X. Hence, M / X is a tg-radical supplementedd. □

Theorem 2.17. Let M be a distributive tg-radical supplemented module. Then, f0r any submodule A 0f M, M/A is tg-radical supplemented.

Proof. Assume that B be a g-supplement submodule in M. At that time there is $B' \leq M$ where

$$M = B + B' \text{ plus } B \cap B' \ll_g B.$$

Since $M = B + B'$, we can write

$$M/A = (B + A)/A + (B' + A)/A.$$

Further, since M is distributive, we get $A + (B \cap B') = (A + B) \cap (A + B')$. This implies that

$$(B + A)/A \cap (B' + A)/A = [(B + A) \cap (B' + A)]/A = \left(A + (B \cap B')\right)/A.$$

Note that $B \cap B' \ll_g B$. Then, we find

$$(B + A)/A \cap (B' + A)/A = (B \cap B' + A)/A \ll_g (B + A)/A.$$

Hence, for any g-supplemen submodule B in M, $(B + A) / A$ is a g-supplemen submodul in M/A. Then, by Lemm 2.16. M/A is tg-radical supplementd. □

3. Cofinitely tg-Radical Supplemented Modules

Definition 3.1. Let M be an R-module. The modul M is said to be cofinitely tg-radical supplemented module if for any cofinite submodule of M has a g-radical supplement which is a supplement in M.

Obviously, all c0finitely ⊕-g-Rad-supplementd moduls are c0finitely tg-radical supplementd.

Lemma 3.2. Suppose that M is a finitely generated modulee. At that time M is tg-radical supplemented if and 0nly if M is cofinitely tg-radical supplemented.

Proof. As M iss finitelly generatedd, thee pr00f is obvious. □

Lemma 3.3. Supposing that M is an R-m0dule besides presume that $Rad_g(M) \leq M$. At that time M is cofinitely tg-radical supplementd if and 0nly if M is cofinitely g-supplementd.

Proof. (⟹) Supposing that A iis any c0finite submodule of M. As M is c0finitely tg-radical supplementd. there existss $B \leq M$ such thatt

$M = A + B$, besides $A \cap B \subseteq Rad_g B$ where B is a g-supplement in M.

Since $A \cap B \subseteq Rad_g\ B \subseteq Rad_g\ M$ and $Rad_g\ M \ll_g M$ we have

$$A \cap B \ll_g M.$$

Hence, we get

$$A \cap B \ll_g B.$$

Then, B is aa g-supplemen 0f A in M. Thus, M is c0finitely g-supplementd.

(⟸) As M is cofinitely g-supplemented, f0r any cofinite submodul A 0f M, there exists $B \leq M$ with

$$M = A + B \qquad \text{besides} \qquad A \cap B \ll_g B.$$

It foll0ws from $A \cap B \subseteq Rad_g B$ that B is a g-radical supplementt 0f A in M. Thenn, M iis cofinitely tg-radical supplemented. □

Lemmaa 3.3 yields the next cor0llary.

Corollary 3.4. Presume that M is a finitely generated R-modul. At that time M is cofinitely tg-radical supplementd if and only if M is c0finitely g-supplementd.

Lemma 3.5. Suppose that M be an R-module and assume that N and G be subm0dules 0f M. If $N + G$ has a g-radical supplement A in M and $N \cap (G + A)$ has a g-radical supplement B in N, then $A + B$ is a g-radical supplement of $G\ in\ M$.

Proof. Assume that A be aa g-radical supplemen of $N + G$ iin M. Then $M = (N + G) + A$ besides $(N + G) \cap A \subseteq Rad_g(A)$. Since $N \cap (G + A)$ has aa g-radical supplemen B iin N, we have

$N = N \cap (G + A) + B$ and $(G + A) \cap B \subseteq Rad_g(B)$. Then $M = N + G + A = [N \cap (G + A) + B] + G + A = G + (A + B)$ besides

$$G \cap (A + B) \leq A \cap (G + B) + B \cap (G + A)$$
$$\leq A \cap (G + N) + B \cap (G + A)$$
$$\leq Rad_g(A) + Rad_g(B)$$
$$\leq Rad_g(A + B).$$

Henceforth $A + B$ is a g-radical supplement of G in M. □

Corolary 2.5 t0gether with Lemmaa 2.3, imply the next chief the0rem:

Theorem 3.6. Every ring R, any direct sum 0f cofinitely tg-radical supplemented R-m0dules is cofinitely tg-radical supplemented.

Proof. Assume that $\{M_i\}_{i\in I}$ bee anyy c0llection 0f c0finitely tg-radical supplemented R-moduls plus suppose that

$$M = \oplus_{i\in I} M_i.$$

Besides suppose that G be any cofinite submodul 0f M. In thiss case, M/G is finitely generatedd besides theree exists $k \in \mathbb{Z}^+$, $x_i \in M, 1 \leq i \leq k,$ suchh thatt

$$M/G = < \{x_1 + G, x_2 + G, \ldots, x_k + G\} >.$$

Hence,

$$M = Rx_1 + Rx_2 + \ldots + Rx_k + G.$$

Then, there exists a finite subset $H = \{i_1, i_{2,} \ldots, i_n\}$ 0f I suchh thatt $x_i \in \oplus_{j\in H} M_j$ f0r any 1$\leq i \leq k$.Then, it is obvious that $M = M_{i1} + \left(G + \sum_{j=2}^{n} M_{ij}\right)$ has aa trivial g-radical supplementt 0 in M.

Assume that a submodulee

$$M_{i1} \cap \left(G + \sum_{j=2}^{n} M_{ij}\right) \leq M_{i1}.$$

Since

$$M_{i1} \,/\, \left[M_{i1} \cap \left(G + \sum_{j=2}^{n} M_{ij}\right)\right] \cong M \,/\, \left(G + \sum_{j=2}^{n} M_{ij}\right) \cong (M/G) \,/\, (\left(G + \sum_{j=2}^{n} M_{ij}\right)/G),$$

we conclude that $M_{i1} \cap \left(G + \sum_{j=2}^{n} M_{ij}\right)$ is a cofinit submodul 0f M_{i1}. In view 0f the fact that M_{i1} is c0finitely tg-radical supplementd, the submodul $M_{i1} \cap \left(G + \sum_{j=2}^{n} M_{ij}\right)$ has a g-radical supplementt V_{i1} such thatt V_{i1} is a g-supplementt in M_{i1}. Using Lemma 3.5, V_{i1} is a g-radical supplementt 0f $G + \sum_{j=2}^{n} M_{ij}$ in M. Likewise, one can sho0w that, for $1 \leq j \leq n, G$ has a g-radical supplementt $V_{i1} + V_{i2} + \cdots + V_{in}$ such that V_{ij} is a suplement in M_{ij} . In this casee, using Corolary 2.5, $V_{i1} + V_{i2} + \cdots + V_{in}$ is a supplement in $M_{i1} \oplus M_{i2} \oplus \cdots \oplus M_{in}$. Since $M_{i1} \oplus M_{i2} \oplus \cdots \oplus M_{in}$ is a direct summand 0f M, via Lemma 2.3, $V_{i1} + V_{i2} + \cdots + V_{in}$ is a g-supplement in M. As a result, $\oplus_{i\in I} M_i$ is cofinitely tg-radical supplementd. □

Definition 3.7. [7] A projective module G is said to be a projective g-cover of a module M if there exists an epimorphism $f: G \to M$ with $Kerf$ is g-small in G, and a module M is called g-semiperfect if every factor module of M has a projective g-cover.

Similar to [11, Lemma 2.2] we have the next lemma.

Lemma 3.8. Suppose that M be a projective module. Let the next conditions:

(i) M is a g-semiperfect module,
(ii) M is ⊕-g-Rad-supplemented module.

Then (i) ⟹ (ii) h0lds. In addition, if M is finitely generated modul, then (ii) ⟹ (i) is too true.

Proof. (i) ⇒ (ii). Let N be a submodul 0f M. Now by supposition, there exists a pr0jective g-c0ver $\pi: P \to M/N$. For the canonical epimorphism $\sigma: M \to M/N$, since M is projective, there is a h0momorphism $f: M \to P$ where $\pi \circ f = \sigma$.

Since π is g-small, f is epic by [2, Prop. 3.2] besides as a result f splits (P is projective). Then, via [2], there exists some homomorphism $g: P \to M$ such that $f \circ g = 1_P$, and hence $\pi = \pi \circ f \circ g = \sigma \circ g$. Note that $M = Kerf \oplus g(p)$ and Ker$f \leq N$; therefore, $M = N + g(p)$. Let μ be the restriction of σ to $g(p)$. Then $\pi = \mu \circ g$ and so μ is epic. As a result since π is g-small, μ is g-small by [2]. That is, $Ker\mu = N \cap g(p) \ll_g g(p)$. Henceforth, $g(p)$ is a g-supplement of N.

(ii) ⇒ (i) Let M be a finitely generatedd modul and N be a submodul to M. As M is ⊕-g-Rad-supplementd modul, there exist submoduls K besides K' of M suchh that $M = N + K$, $N \cap K \leq Rad_g(K)$, besides $K \oplus K' = M$. Clearly, K is projective. F0r the inclusi0n homom0rphism $i: K \to M$ besides the canonical epimorphis $\sigma: M \to M/N$, $\sigma \circ i: K \to M/N$ is an epimorphism, and by supposition $Rad_g M \ll_g M$, this implies that $Rad_g K \ll_g K$ besides henceforth $Ker(\sigma \circ i) = N \cap K \ll_g K$. □

Theorem 3.9. Presume M is a projective besides finitely generated R-modul. At that time the next assertions are equivalent:

(i) M is a g-semiperfect module;
(ii) M is ⊕-g-Rad-supplemented;
(iii) M is cofinitely ⊕-g-Rad-supplemented;
(iv) M is tg-radical supplemented;
(v) M is cofinitely tg-radical supplemented.

Proof. (i)⟺(ii) obvious using Lemmaa 3.8.

(ii)⟹(iv) Immediately foll0ws fr0m the definiti0ns.

(iv)⟹(ii) As M is tg-radical supplementd plus finitely generatedd, by Lem 2.7, M is g-supplementd. Then again, as M is projective, one concludes that M is ⊕-g-supplemented. As a result, M is ⊕-g-Rad-supplemented. As M iis finitely generatedd, itt is obvious thatt (ii) ⟺ (iii) besides (iv)⟺(v).

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