

On the Application of the Emad-Sara Transform Technique to the Solution of Complex Differential Equations

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Abstract. When attempting to express a differential equation of the first order, it is usual practice to use the equation represented by $dy/dx = f(x,y)$. The above equation contains a function denoted by the notation embodied by $f(x,y)$, defined on a portion of the xy -plane and dependent on two variables, x , and y , which are independent variables. The equation considered to be of the first order is the one that has just the first derivative, which is denoted by the notation dy/dx since there are no higher-order derivatives present in the equation. It is usual practice to make use of the differential equation when attempting to explain the connection that exists between a function and the derivatives of that function. Using this technique to identify functions inside a given domain in physics, chemistry, and many other fields of science is feasible. In order to do so, it is necessary to have previous knowledge about functions and the derivatives of those functions. In this article, the Emad-Sara transformation is presented as a method for identifying the general solution of complex differential equations of the first order that have constant coefficients. This transformation may be used to get the general solution of these equations. This approach, which is both practical and economical, may be used to find solutions to a wide variety of linear operator equations. These equations can range from simple to complex

1. Introduction

It is possible to convert complex differential equations into partial differential equations (PDEs) comprising two sets of unknowns. (Two independent variables). The process of isolating the real and imaginary components of a complex number. Determining the order of a partial differential equation is based on identifying the highest derivative present in the equation. The specific solution to a partial differential equation refers to a function that satisfies the equation and renders it an identity upon substitution. A solution is general if it encompasses all specific solutions of the relevant equation. The nomenclature "exact solution" is frequently employed in nonlinear partial differential equations of second and higher orders[1,2]. Partial differential equations (PDEs) are frequently employed in resolving various issues, including but not limited to the transmission of heat or sound, fluid dynamics, elasticity, electrostatics, and electrodynamics[3].

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The Emad-Sara transformation method has been employed to address many problems in various mathematical domains[4,5]. Employing an integral transform is a viable approach for resolving linear ordinary and partial differential equations[6,7,8]. Because of nonlinearity terms, the Emad-Sara integral transform in isolation is insufficient for determining the solution of nonlinear differential equations. In conjunction with the differential transformation technique, the Emad-Sara integral transformation presents a viable approach for resolving nonlinear differential equations.

The present study investigates the utilization of the Emad-Sara transformation of first-order complex differential equations with constant coefficients. The paper introduces solutions and explores the transform's efficacy in performing tasks previously accomplished by the Laplace transform, and the SEE transform, as documented in references.

2. Basic Concepts

The definition of the Emad-Sara integral transformation of the function $f(t)$, $\forall t \geq 0$ is [6]:

$$ES\{f(t)\} = T(v) = \frac{1}{v^2} \int_{t=0}^{\infty} e^{-vt} f(t) dt. \tag{1}$$

where $L_1 \leq V \leq L_2$ and $L_1, L_2 > 0$ for all $t \geq 0$.

Emad-Sara transformation operator is $ES \{.\}$.

Table 1 shows the application of the Emad-Sara transformation to some functions.

Table 1: The Emad-Sara transformation application on some fundamental functions⁶.

$f(t)$	$ES\{f(t)\} = T(v)$
$k \equiv \text{constant.}$	$\frac{k}{v^3}.$
t^n, n is a positive integer number.	$\frac{n!}{v^{n+3}}.$
e^{at}, a is a constant.	$\frac{1}{v^2(v-a)}.$
$\text{Sin}(at).$	$\frac{a}{v^2(v^2+a^2)}.$
$\text{cos}(at).$	$\frac{1}{v(v^2+a^2)}.$
$\text{sinh}(at).$	$\frac{a}{v^2(v^2-a^2)}.$
$\text{cosh}(at).$	$\frac{1}{v(v^2-a^2)}.$
$f(at)$ (Change of scale property)	$\frac{1}{a^3} T\left(\frac{v}{a}\right).$
$e^{at}f(t)$ (Shifting property)	$\frac{(v-a)^2}{v^2} T(v-a).$
$t f(t).$	$\frac{-2}{v} T(v) - \frac{dT(v)}{dv}.$

Theorem 1⁶

Let $T(v)$ be the Emad-Sara transformation of $(ES\{f(t)\} = T(v))$ then:

- $ES\{f'(t)\} = -\frac{1}{v^2} f(0) + vT(v).$
- $ES\{f''(t)\} = \frac{-f'(0)}{v^2} - \frac{f(0)}{v} + v^2 T(v).$
- In general: $ES\{f^{(n)}(t)\} = -\frac{f^{(n-1)}(0)}{v^2} - \frac{f^{(n-2)}(0)}{v} - \dots - \frac{f(0)}{v^{3-n}} + v^n T(v).$

Theorem 2⁶

Emad-Sara integral transform of partial derivatives in the form $ES\{f(x, t)\} = T(x, v)$ are:

- $ES\{f_t\} = vT(x, v) - \frac{1}{v^2} f(x, 0).$

2. $ES\{f_x\} = \frac{d}{dx} [T(x, v)] .$
3. $ES\{f_{xx}\} = \frac{d^2}{dx^2} [T(x, v)] .$
4. $ES\{f_{tt}\} = v^2 T(x, v) - \frac{1}{v^2} \frac{\partial f}{\partial x}(x, 0) - \frac{f(x, 0)}{v} .$

Complex Derivatives Definition[10]

If $w = w(z, \bar{z})$ is a complex function, where $z = x + iy$ and $w(z, \bar{z}) = p(x, y) + iq(x, y)$. Then the first order derivatives according to z and \bar{z} of $w(z, \bar{z})$ are defined as following:

$$w_z = \frac{1}{2}(w_x - iw_y), \tag{2}$$

$$w_{\bar{z}} = \frac{1}{2}(w_x + iw_y). \tag{3}$$

Utilizing Of the Emad-Sara Transform Technique for Solving First Order Complex Differential Equations with Constant Coefficients

Theorem 3

Suppose that a, b and c are real constants, $F(z, \bar{z})$ is a polynomial of z, \bar{z} and $w = p + iq$ is a complex function. Then the real and imaginary parts of solution of:

$$\left. \begin{aligned} a \frac{\partial w}{\partial z} + b \frac{\partial w}{\partial \bar{z}} + cw &= F(z, \bar{z}), \\ \text{and } \omega(x, 0) &= 0. \end{aligned} \right\} \tag{4}$$

Are

$$p = (ES)^{-1} \left\{ \frac{(a + b) \frac{\partial}{\partial x} \left[2E_3 + \frac{(a - b)}{v^2} q(x, 0) \right]}{\Delta} \right\} + (ES)^{-1} \left\{ \frac{2C \left(2E_3 + \frac{(a - b)}{v^2} q(x, 0) \right) - (a - b)v \left(4E_4 + \frac{(b - a)}{v^2} p(x, 0) \right)}{\Delta} \right\} .$$

And,

$$q = (ES)^{-1} \left\{ \frac{(a + b) \frac{\partial}{\partial x} \left[2E_4 + \frac{(a - b)}{v^2} p(x, 0) \right]}{\Delta} \right\} + (ES)^{-1} \left\{ \frac{2C \left(2E_4 + \frac{(a - b)}{v^2} p(x, 0) \right) - (a - b)v \left(2E_3 + \frac{(a - b)}{v^2} q(x, 0) \right)}{\Delta} \right\} .$$

Where:

$$\Delta = \begin{vmatrix} (a + b)D + 2C & (a - b)v \\ (b - a)v & (a + b)D + 2C \end{vmatrix} = ((a + b)D + 2C)^2 + v^2(b - a)^2 .$$

And E_1, E_2, E_3 and E_4 are Emad-Sara transform of p, q, F_1 and F_2 respectively.

Proof: Using Eq. 2 and Eq. 3 in Eq. 4, gives:

$$\frac{a}{2}(w_x - iw_y) + \frac{b}{2}(w_x + iw_y) + cw = F_1(x, y) + iF_2(x, y). \tag{5}$$

Applying $w = p + iq$ in Eq. 5, then the following Eq. is obtained:

$$a(p_x + iq_x - ip_y + p_y) + b(p_x + iq_x + ip_y - q_y) + 2cw = 2F_1 + 2iF_2. \tag{6}$$

Upon separating Eq. 6 into its real and imaginary components, the ensuing system is derived:

$$(a + b)p_x + (a - b)q_y + 2cp = 2F_1, \text{ (Real part)} \tag{7}$$

$$(a + b)q_x + (b - a)p_y + 2cq = 2F_2, \text{ (Imaginary part)}. \tag{8}$$

Upon application of the Emad-Sara transformation technique to the aforementioned Eq. 7 and Eq. 8, the resulting expressions are as follows:

$$(a + b) \frac{\partial E_1}{\partial x} + (a - b) \left[vE_2 - \frac{1}{v^2} q(x, 0) \right] + 2CE_1 = 2E_3, \tag{9}$$

$$(a + b) \frac{\partial E_2}{\partial x} + (b - a) \left[vE_1 - \frac{1}{v^2} p(x, 0) \right] + 2CE_2 = 2E_4. \tag{10}$$

If Eq. 9 and Eq. 10 are properly regulated and subjected to Cramer's rule, the resulting equations can be obtained:

$$(a + b) \frac{\partial E_1}{\partial x} + (a - b)vE_2 + 2CE_1 = 2E_3 + \frac{(a - b)}{v^2} q(x, 0),$$

$$(b - a)vE_1 + (a + b) \frac{\partial E_2}{\partial x} + 2CE_2 = 2E_4 + \frac{(b - a)}{v^2} p(x, 0).$$

Now

$$E_1 = \frac{\begin{vmatrix} 2E_3 + \frac{(a - b)}{v^2} q(x, 0) & (a - b)v \\ 2E_4 + \frac{(b - a)}{v^2} p(x, 0) & (a + b)D + 2C \end{vmatrix}}{\Delta},$$

So,

$$E_1 = \frac{(a + b) \frac{\partial}{\partial x} \left[2E_3 + \frac{(a - b)}{v^2} q(x, 0) \right]}{\Delta} + \frac{2C \left(2E_3 + \frac{(a - b)}{v^2} q(x, 0) - (a - b)v \left(2E_4 + \frac{(b - a)}{v^2} p(x, 0) \right) \right)}{\Delta}, \tag{11}$$

And,

$$\frac{\begin{vmatrix} (a + b)D + 2C & 2S_3 + \frac{(a - b)}{v^2} q(x, 0) \\ (a - b)v & 2S_4 + \frac{(b - a)}{v^2} p(x, 0) \end{vmatrix}}{\Delta},$$

So,

$$E_2 = \frac{(a + b) \frac{\partial}{\partial x} \left[2E_4 + \frac{(b - a)}{v^2} p(x, 0) \right]}{\Delta} + \frac{2C \left(2E_4 + \frac{(b - a)}{v^2} p(x, 0) - (b - a)v \left(2E_3 + \frac{(a - b)}{v^2} q(x, 0) \right) \right)}{\Delta}. \tag{12}$$

The obtained results are derived using the inverse Emad-Sara transform method applied to Eq. 11 and Eq. 12.

$$p(x, y) = (ES)^{-1} \left\{ \frac{(a+b) \frac{\partial}{\partial x} \left[2E_3 + \frac{(a-b)}{v^2} q(x, 0) \right]}{\Delta} \right\} + (ES)^{-1} \left\{ \frac{2C \left(2E_3 + \frac{(a-b)}{v^2} q(x, 0) \right) - (a-b)v \left(2E_4 + \frac{(b-a)}{v^2} p(x, 0) \right)}{\Delta} \right\}, \tag{13}$$

And,

$$q(x, y) = (ES)^{-1} \left\{ \frac{(a+b) \frac{\partial}{\partial x} \left[2E_4 + \frac{(b-a)}{v^2} p(x, 0) \right]}{\Delta} \right\} + (ES)^{-1} \left\{ \frac{2C \left(2E_4 + \frac{(b-a)}{v^2} p(x, 0) \right) - (b-a)v \left(2E_3 + \frac{(a-b)}{v^2} q(x, 0) \right)}{\Delta} \right\}. \tag{14}$$

Application

Consider the following complex differential Eq.:

$$3 \frac{\partial w}{\partial z} + \frac{\partial w}{\partial \bar{z}} = 0,$$

with partial condition:

$$w(x, 0) = x^2.$$

The above coefficients of Eq. are $a = 3, b = 1$ and $c = 0$, also $F(x, y) = 0$.

From the above theorem: $\Delta = 16D^2 + 4v^2$.

And,

$$\begin{aligned}
 p &= (ES)^{-1} \left\{ \frac{-2v \left(\frac{-2}{v^2} x^2 \right)}{16D^2 + 4v^2} \right\} = 4x^2(ES)^{-1} \left\{ \frac{\frac{1}{v}}{16D^2 + 4v^2} \right\}, \\
 p &= x^2(ES)^{-1} \left\{ \frac{\frac{1}{v}}{4D^2 + v^2} \right\} = x^2(ES)^{-1} \left\{ \frac{\frac{1}{v}}{v^2 \left(\frac{4D^2}{v^2} + 1 \right)} \right\}, \\
 p &= x^2(ES)^{-1} \left\{ \frac{1}{v^3} \left(1 - \left(\frac{2D}{v} \right)^2 \right) \right\}, \\
 p &= (ES)^{-1} \left\{ \frac{x^2}{v^3} \left(1 - 2 \left(\frac{2}{v} \right)^2 \right) \right\}, \\
 p(x, y) &= (ES)^{-1} \left\{ \frac{x^2}{v^3} \right\} - (ES)^{-1} \left\{ \frac{8x^2}{v^5} \right\}, \\
 p(x, y) &= x^2 - 4x^2y^2.
 \end{aligned}$$

On the other hand:

$$\begin{aligned}
 q &= (ES)^{-1} \left\{ 4 \frac{\partial}{\partial x} \left(\frac{\frac{-2}{v^2} \cdot x^2}{16D^2 + 4v^2} \right) \right\} = -16x(ES)^{-1} \left\{ \frac{1}{v^2(16D^2 + 4v^2)} \right\}, \\
 q(x, y) &= -4x(ES)^{-1} \left\{ \frac{1}{v^2(4D^2 + v^2)} \right\} = -4x(ES)^{-1} \left\{ \frac{1}{v^4 \left(1 + \left(\frac{2D}{v} \right)^2 \right)} \right\}, \\
 q(x, y) &= -4x(ES)^{-1} \left\{ \frac{1}{v^4} \left(1 - \left(\frac{2D}{v} \right)^2 \right) \right\}, \\
 q(x, y) &= -4(ES)^{-1} \left\{ \frac{x}{v^4} (1) \right\} = -4xy.
 \end{aligned}$$

So,

$$\begin{aligned}
 W &= p(x, y) + iq(x, y), \\
 W &= x^2 - 4x^2y^2 - 4ixy.
 \end{aligned}$$

3. Conclusions and future directions

This paper presents an investigation into the utilization of the Emad-Sara transform technique for determining the comprehensive, exact solution of first-order complex differential equations with constant coefficients, along with its primary applications. When used in tandem with the differential transformation technique, the Emad-Sara integral transform has demonstrated its efficacy in addressing nonlinear differential equations; this has been demonstrated by applying the transform in two such scenarios.

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