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Characterizations of Paracompact Map

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Abstract. Paracompactness is a tool for splitting of a space. In this paper, new types of maps initiated called a paracompact map, S-paracompact mapping and β-paracompact mapping. The properties of these maps investigated. In addition, the relationships between a paracompact map with other maps discussed, such as Bourbaki proper mapping and closed mapping. Moreover, we studied the necessary conditions and properties for satisfying the composition of the new maps with the projection map and constant map are paracompact. The product of two paracompact maps also verified under necessary conditions

1. INTRODUCTION

The term of a paracompact space initiated in 1944 by Dieudonné [1]. In 1948, Stone [2] showed that this term is indistinguishable with the concept of full normality which introduced in 1940 by Tuke [3]. Consequently, a new type of paracompactness presented in 2006 by Al-Zoubi namely S-paracompact spaces [4]. Then, in 2013, Demir and Ozbakir initiated β-paracompact space [5]. On the other side, certain types of maps introduced by using the concept of compactness as a compact map which originated in 1957 by Halfar [6]. in 1989, Bourbaki [7] invented a proper map and he gave various properties of this map.

2. PRILIMINARIES

This section includes all the basic definitions and theorems that needed in this work.

Definition 2.1 [1] A Hausdorff space $\mathbb{W}$ is said to be paracompact if any open cover of it has a locally-finite open refinement.

Definition 2.2 [8] A space $\mathbb{W}$ is said to be extremally disconnected (briefly e.d.) if the closure of every open set in $\mathbb{W}$ is open.

Definition 2.3 [9] A space $\mathbb{W}$ is called submaximal if each dense subset of $\mathbb{W}$ is open in $\mathbb{W}$.

Remark 2.4.[5]

(i) Every paracompact space is β-paracompact.
(ii) Every S-paracompact space is β-paracompact.

Theorem 2.5. [5] Let $\mathbb{W}$ be an e.d. submaximal space. If $\mathbb{W}$ is β-paracompact, then $\mathbb{W}$ is paracompact.

Definition 2.6. [7] Let $\mathbb{W}$ and $\mathbb{M}$ be two spaces. A map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is said to be proper if:

- $\mathcal{L}$ is a continuous map.
- $\mathcal{L} \times I_{\mathbb{E}}: \mathbb{W} \times \mathbb{E} \to \mathbb{M} \times \mathbb{E}$ is closed, for any space $\mathbb{E}$.

Theorem 2.7. [7] Let $\mathbb{W}$ and $\mathbb{M}$ be spaces and $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is a continuous map. Then, the following statements are equivalent:
(i) $\mathcal{L}$ is a proper map.
(ii) $\mathcal{L}$ is a closed map and $\mathcal{L}^{-1}(\{y\})$ is compact for each $y \in \mathbb{M}$.
(iii) If (χ_d) are a net in $\mathbb{W}$ and $y \in \mathbb{M}$ is a cluster point of the net $\mathcal{L}(\chi_d)$, then there is a cluster point $x \in \mathbb{W}$ of (χ_d), such that $\mathcal{L}(x) = y$.

Proposition 2.8. [7] Let $\mathcal{L}_1: \mathbb{W}_1 \to \mathbb{M}_1$ and $\mathcal{L}_2: \mathbb{W}_2 \to \mathbb{M}_2$ maps, then $\mathcal{L}_1 \times \mathcal{L}_2: \mathbb{W}_1 \times \mathbb{W}_2 \to \mathbb{M}_1 \times \mathbb{M}_2$ is proper if and only if $\mathcal{L}_1$and $\mathcal{L}_2$are proper.

Lemma 2.10. [10] A continuous map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is closed iff for any point $y \subseteq \mathbb{M}$, and any open set $U \subseteq \mathbb{W}$ which contains $\mathcal{L}^{-1}(\{y\})$, then there is neighborhood set V of y in a space $\mathbb{M}$ such that $\mathcal{L}^{-1}(V) \subseteq U$.

Proposition 2.11. [7] Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a proper map. If H is a subset of $\mathbb{M}$, then $\mathcal{L}_H: \mathcal{L}^{-1}(H) \to H$ is proper.

Proposition 2.12. [7] If $\mathcal{J} o \mathcal{L}$ is proper and $\mathcal{J}$ is injective open, then a continuous map $\mathcal{L}$ is proper.

Proposition 2.13. [5] The topological product $\mathbb{W} \times \mathbb{M}$ of compact space $\mathbb{W}$ and a paracompact space $\mathbb{M}$ is paracompact.

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3. MAIN RESULTS

Definition 3.1. A surjective continuous map $\mathcal{L}\colon \mathbb{W} \to \mathbb{M}$ is said to be paracompact if the inverse image for any paracompact set in $\mathbb{M}$ is paracompact set in $\mathbb{W}$.

Proposition 3.2. A continuous map $\mathcal{L}$ from $\mathbb{W}$ onto $\mathbb{M}$ is paracompact, if and only if every $y \in \mathbb{M}$ and every family $\mathbb{U} = \{U_\alpha\}_{\alpha\in I}$ of open subsets of $\mathbb{W}$ satisfying $\mathcal{L}^{-1}(\{y\}) \subseteq \cup\, \mathbb{U}$, there exists a neighborhood V_y of y such that $\mathcal{L}^{-1}(\{y\})$ is covered by $\mathbb{U}$ and $\{U_\alpha \cap \mathcal{L}^{-1}(V_y)\}_{\alpha\in I}$ has an open refinement $\mathbb{V}$ such that $\mathbb{V}$ is locally finite at $\mathcal{L}^{-1}(\{y\})$.

Proof. Assume that $y \in \mathbb{M}$ and $\mathbb{U} = \{U_\alpha\}_{\alpha\in I}$ is an open cover of $\mathcal{L}^{-1}(\{y\})$, that is $\mathcal{L}^{-1}(\{y\}) \subseteq \cup\, \mathbb{U}$. Choose $\mathbb{M}$ as the open set containing y. Since $\{y\}$ is paracompact set in $\mathbb{M}$ and $\mathcal{L}$ is paracompact map, then $\mathcal{L}^{-1}(\{y\})$ is paracompact set in $\mathbb{W}$, and so the open cover $\{U_\alpha \cap \mathbb{W}\}_{\alpha\in I} = \mathbb{U}$ of $\mathcal{L}^{-1}(\{y\})$ has open refinement $\mathbb{V}$ which is locally refinement at $\mathcal{L}^{-1}(\{y\})$. Conversely, let $K \subseteq \mathbb{M}$ be paracompact. For every $y \in K$ and every open cover $\mathbb{U} = \{U_\alpha\}_{\alpha\in I}$ of $\mathcal{L}^{-1}(K)$ such that $\mathcal{L}^{-1}(\{y\}) \subseteq \cup\, \mathbb{U}$, then there exists a neighborhood V_y of y and $\mathcal{L}^{-1}(V_y) \subseteq \cup\, \mathbb{U}$ also, $\mathcal{L}^{-1} \cup V_y \subseteq \cup\, \mathbb{U}$. Since $\mathcal{L}^{-1}(K) \subseteq \mathcal{L}^{-1} \cup V_y$, then $\{U_\alpha \cap \mathcal{L}^{-1} \cup V_y\}_{\alpha\in I}$ is cover of $\mathcal{L}^{-1}(K)$ and it has open refinement and so $\mathbb{U}$ has open refinement which is locally finite at $\mathcal{L}^{-1}(K)$.

Definition 3.3. A surjective continuous map $\mathcal{L}\colon \mathbb{W} \to \mathbb{M}$ is said to be S-paracompact (β-paracompact) if the inverse image for any S-paracompact (β-paracompact) set in $\mathbb{M}$ is S-paracompact(β-paracompact) set in $\mathbb{W}$.

Proposition 3.4. If $\mathcal{L}\colon \mathbb{W} \to \mathbb{M}$ is a bijective continuous map, then the following statements are equivalent:

(i) $\mathcal{L}$ is a proper map.
(ii) $\mathcal{L}$ is a paracompact map.
(iii) $\mathcal{L}$ is a closed map.
(iv) $\mathcal{L}$ is a homeomorphism of $\mathbb{W}$ on to closed subset of $\mathbb{M}$.

Proof. (i→ii) Suppose that $\mathcal{L}\colon \mathbb{W} \to \mathbb{M}$ is a proper map and $K \subseteq \mathbb{M}$ is a paracompact set in $\mathbb{M}$. For any open cover $\mathbb{U} = \{U_\alpha\}_{\alpha\in I}$ of $\mathcal{L}^{-1}(K)$ and $y \in K$, thus there is a finite subcover $\mathbb{U}_y \subseteq \mathbb{U}$ s.t. $\mathcal{L}^{-1}(\{y\}) \subseteq \cup\, \mathbb{U}_y$. Now, take an open neighborhood V_y of y such that $\mathcal{L}^{-1}(V_y) \subseteq \cup\, \mathbb{U}_y$. Since Kis paracompact, then the open cover $\{V_y : y \in K\}$ of K has locally finite open refinement $\{S_y : y \in K\}$. Thus $\{\mathcal{L}^{-1}(S_y) \cap U : y \in K, U \in \mathbb{U}_y\}$ is a locally finite open refinement of $\mathbb{U}$. Hence $\mathcal{L}^{-1}(K)$ is paracompact in $\mathbb{W}$.

(ii→iii) Let $F \subseteq \mathbb{M}$ and U be an open set in $\mathbb{W}$ such that $\mathcal{L}^{-1}(F) \subseteq U$, then $\mathcal{L}^{-1}(\{y\}) \subseteq U$ for all $y \in F$. Since $\mathcal{L}$ is a paracompact map, then there exists an open neighborhood $V_y$of y and $\{U\}$ has open refinement $\{W_y\}$ such that $\mathcal{L}^{-1}(V_y) \subseteq W_y \subseteq U$ by Proposition (3.2), so $\mathcal{L}^{-1}(\cup\, V_y) \subseteq U$. But $\cup\, V_y$is an open set and $\mathcal{L}$ is continuous, thus $\mathcal{L}^{-1}(\cup\, V_y)$ is an open set contained in U. Therefore, $\mathcal{L}$ is a closed map by Lemma (2.9).

(iii→iv) From (iii) $\mathcal{L}$ is a closed map and by hypothesis, $\mathcal{L}$ is continuous and bijective, so $\mathcal{L}$ is a homeomorphism map.

(iv→i) Assume that $\mathcal{L}$ is the homeomorphism map and $\mathbb{K}$ is any space, then $\mathcal{L} \times I_{\mathbb{K}}\colon \mathbb{W} \times \mathbb{K} \to \mathbb{M} \times \mathbb{K}$ is homeomorphism that is $\mathcal{L} \times I_{\mathbb{K}}$is closed and $\mathcal{L}$ is continuous by hypothesis. Thus, $\mathcal{L}$ is a proper map.

Corollary 3.5. Let $\mathcal{L}\colon \mathbb{W} \to \mathbb{M}$ be a surjective and proper map, then the inverse image for any paracompact set in $\mathbb{W}$ is paracompact set in $\mathbb{M}$.

Theorem 3.6. Paracompactness is a topological property.

Proof. Assume that $\mathcal{L}\colon \mathbb{W} \to \mathbb{M}$ a homeomorphism map and A is paracompact set in $\mathbb{W}$. If $\mathbb{U} = \{U_\alpha\}_{\alpha\in I}$ is an open cover of $\mathcal{L}(A)$, then $\mathcal{L}^{-1}(\mathbb{U})$ is an open cover of A and so, there exists a locally finite open refinement $\mathbb{V}$ of $\mathcal{L}^{-1}(\mathbb{U})$. Since $\mathcal{L}$ is open then $\mathcal{L}(\mathbb{V})$ is an open refinement of $\mathbb{U}$. Now, to prove that $\mathcal{L}(\mathbb{V})$ is locally finite in $\mathcal{L}(A)$. Let $y \in \mathcal{L}(A)$. Since $\mathbb{V}$ is locally finite, then for each $x \in \mathcal{L}^{-1}(\{y\})$ there exists an open neighborhood D_x of x such that D_x intersects at most finitely many members of $\mathbb{V}$ and $\{D_x : x \in \mathcal{L}^{-1}(\{y\})\}$ is an open cover of $\mathcal{L}^{-1}(\{y\})$. Proposition 3.2 implies that $\mathcal{L}$ is a proper map and so there exists a finite subcollection $\mathbb{D}_y$ of $\{D_x\}$ such that $\mathcal{L}^{-1}(\{y\}) \subseteq \cup\, \mathbb{D}_y$, also $\cup\, \mathbb{D}_y$ intersects at most finitely many members of $\mathbb{V}$. Since $\mathcal{L}$ is a continuous and closed map then By Lemma (1.10.), there exists an open neighborhood E_y of y in $\mathcal{L}(A)$ such that $\mathcal{L}^{-1}(E_y) \subseteq \cup\, \mathbb{D}_y$, then $\mathcal{L}^{-1}(E_y)$ intersects at most finitely many members of $\mathbb{V}$, therefore E_y intersects at most finitely many members of $\mathcal{L}(\mathbb{V})$. Thus, $\mathcal{L}(\mathbb{V})$ is locally finite in $\mathcal{L}(A)$.

Corollary 3.7. Let $\mathcal{L}\colon \mathbb{W} \to \mathbb{M}$ be a surjective, open and proper, then the image for any paracompact set in $\mathbb{W}$ is paracompact set in $\mathbb{M}$.

Lemma 3.8. Let $\mathbb{W}$ be a space. If $\mathbb{W} \times \{y\}$ is paracompact then $\mathbb{W}$ is paracompact, for each point y which does not belong to $\mathbb{W}$.

Proof. Let $\mathbb{U} = \{U_\alpha\}_{\alpha\in I}$ be an open cover of $\mathbb{W}$, then $\mathbb{W} \subseteq \cup_{\alpha\in I} U_\alpha$ and this implies $\mathbb{W} \times \{y\} \subseteq \cup_{\alpha\in I} U_\alpha \times \{y\}$. Since $\mathbb{W} \times \{y\}$ is paracompact, then $\{U_\alpha\}_{\alpha\in I} \times \{y\}$ has a locally finite open refinement $\{V_\beta\}_{\beta\in J} \times \{y\}$. Therefore, $\{V_\beta\}_{\beta\in J}$ a locally finite open refinement of $\mathbb{U}$. Hence, $\mathbb{W}$ is paracompact.

Proposition 3.9. Every compact set is paracompact.

The converse is not, in general, true.

Example 3.10. Let $\mathbb{W}$ be a countable space and τ be discrete topology on $\mathbb{W}$ i.e. $\tau = \{\mathbb{W}, \emptyset, \{x: x \in \mathbb{W}\}\}$. If $\mathbb{U} = \{x: x \in \mathbb{W}\}$, then $\mathbb{U}$ is an open cover of $\mathbb{W}$ which has no finite subcover, this means $\mathbb{W}$ is not compact.

On the other hand, for $x \in \mathbb{W}$, $\{x\}$ is an open set containing x. If $\mathbb{V}$ is the collection of open subsets which covers $\mathbb{W}$, then $\mathbb{U}$ is an open refinement of $\mathbb{V}$. Also, $\{x\}$ intersects finitely many members of $\mathbb{U}$and so, $\mathbb{U}$ is locally finite. Hence, $\mathbb{W}$ is a paracompact set.

Proposition 3.11. Let $\mathcal{L}_1: \mathbb{W}_1 \to \mathbb{M}_1$ and $\mathcal{L}_2: \mathbb{W}_2 \to \mathbb{M}_2$ be bijective continuous maps, then $\mathcal{L}_1 \times \mathcal{L}_2: \mathbb{W}_1 \times \mathbb{W}_2 \to \mathbb{M}_1 \times \mathbb{M}_2$ is paracompact if and only if $\mathcal{L}_1$and $\mathcal{L}_2$are paracompact.

Proof. Assume that $\mathcal{L}_1$is bijective continuous. Let $A \subseteq \mathbb{M}_1$ be paracompact in $\mathbb{M}_1$ and let $y_2 \in \mathbb{M}_2$, since $\{y_2\}$ is compact then by Proposition (3.9), $\{y_2\}$ is paracompact. By Proposition (2.13), $A \times \{y_2\}$ is paracompact set in $\mathbb{M}_1 \times \mathbb{M}_2$. Since $\mathcal{L}_1 \times \mathcal{L}_2$ is paracompact, then $(\mathcal{L}_1 \times \mathcal{L}_2)^{-1}(A \times \{y_2\})=\mathcal{L}_1^{-1}(A) \times \mathcal{L}_2^{-1}(\{y_2\})$ is paracompact, by Lemma (3.8), $\mathcal{L}_1^{-1}(A)$ is paracompact. Therefore, $\mathcal{L}_1$ is a paracompact map.

Similarly, to prove that $\mathcal{L}_2$ is a paracompact map.

Conversely, By Proposition (3.4), $\mathcal{L}_1$ and $\mathcal{L}_2$ are proper maps, thus by Proposition (2.8), $\mathcal{L}_1 \times \mathcal{L}_2$ is a proper map. Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are surjective continuous, then $\mathcal{L}_1 \times \mathcal{L}_2$ is surjective continuous. Hence, $\mathcal{L}_1 \times \mathcal{L}_2$ is Paracompact by Proposition (3.4).

Corollary 3.12. Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a paracompact map and $I_{\mathbb{E}}: \mathbb{E} \to \mathbb{E}$, then $\mathcal{L} \times I_{\mathbb{E}}: \mathbb{W} \times \mathbb{E} \to \mathbb{M} \times \mathbb{E}$ is paracompact, for any space $\mathbb{E}$.

Proposition 3.13. Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$, $\mathcal{J}: \mathbb{M} \to \mathbb{E}$ and $\mathcal{J} o \mathcal{L}: \mathbb{W} \to \mathbb{E}$ be a map, then:

(i) If $\mathcal{J}$ and $\mathcal{L}$ are paracompact, then $\mathcal{J} o \mathcal{L}$ is paracompact map.

Proof. Since $\mathcal{L}$ and $\mathcal{J}$ are surjective continuous, then $\mathcal{J} o \mathcal{L}$ is surjective continuous.

Let K be paracompact set in $\mathbb{E}$, then $\mathcal{J}^{-1}(K)$is paracompact set in $\mathbb{M}$, also $\mathcal{L}^{-1}\left(\mathcal{J}^{-1}(K)\right) = (\mathcal{J} o \mathcal{L})^{-1}(K)$ is paracompact in $\mathbb{W}$. Therefore, $\mathcal{J} o \mathcal{L}$ is paracompact.

(ii) If $\mathcal{L}$ is paracompact and $\mathcal{J}$ is a homeomorphism, then $\mathcal{J} o \mathcal{L}$ is paracompact.

Proof. Since $\mathcal{J}$ is a closed map, then $\mathcal{J}$ is homeomorphism on a closed subset of . Consequently, $\mathcal{J}$ is paracompact by Proposition (3.4), thus by (i), $\mathcal{J} o \mathcal{L}$ is paracompact.

(iii) If $\mathcal{J}$ is continuous and compact and $\mathcal{L}$ is paracompact, then $\mathcal{J} o \mathcal{L}$ is a compact, where $\mathbb{W}$ is countable compact and $\mathbb{E}$ is Hausdorff space.

Proof. Let K be a compact set in $\mathbb{E}$, then K a closed set. Consequently, $\mathcal{J}^{-1}(K)$ is closed compact set in $\mathbb{M}$, $\mathcal{J}^{-1}(K)$ is closed paracompact set by Proposition (3.9). Since $\mathcal{L}$ is paracompact and $\mathbb{W}$ is countable compact, then $\mathcal{L}^{-1}\left(\mathcal{J}^{-1}(K)\right) = (\mathcal{J} o \mathcal{L})^{-1}(K)$ is compact in $\mathbb{W}$. Hence, $\mathcal{J} o \mathcal{L}$ is compact.

(iv) If $\mathcal{J} o \mathcal{L}$ is paracompact and $\mathcal{L}$ is surjective open proper, then $\mathcal{J}$ is paracompact map.

Proof. Since $\mathcal{J} o \mathcal{L}$ is surjective then $\mathcal{J}$ is surjective. Let B is an open set in $\mathbb{E}$, then $\mathcal{L}^{-1} o \mathcal{J}^{-1}(B)$ is open in $\mathbb{W}$ and so, $\mathcal{L}(\mathcal{L}^{-1} o \mathcal{J}^{-1}(B)) = \mathcal{J}^{-1}(B)$ is open in $\mathbb{M}$. Thus, $\mathcal{J}$ is continuous.

Suppose that $\mathcal{J} o \mathcal{L}$ is paracompact map and K is paracompact set in $\mathbb{E}$, then $\mathcal{L}^{-1} o \mathcal{J}^{-1}(K)$ is paracompact in $\mathbb{W}$. Since $\mathcal{L}$ is open proper then, $\mathcal{L}(\mathcal{L}^{-1} o \mathcal{J}^{-1}(K)) = \mathcal{J}^{-1}(K)$ is paracompact in $\mathbb{M}$ by Corollary (3.7). Hence, $\mathcal{J}$ is paracompact map.

(v) If $\mathcal{L}$ is β-paracompact map and $\mathbb{W}$ is an e.d. submaximal space, then $\mathcal{L}$ is paracompact.

Proof. Let $K \subseteq \mathbb{M}$ be a paracompac set. Then by Remark (1.8.), K is β-paracompact in $\mathbb{M}$. But, $\mathcal{L}$ is β-paracompact map, then $\mathcal{L}^{-1}(K)$ is β-paracompact set. By Theorem 2.5, $\mathcal{L}^{-1}(K)$ is paracompact in $\mathbb{W}$. Hence, $\mathcal{L}$ is paracom-pact.

(vi) If $\mathcal{L}$ is β-paracompact and $\mathcal{J}$ is an S-paracompact map, then $\mathcal{J} o \mathcal{L}$ is paracompact, where $\mathbb{W}$ is an e.d. submaximal space.

Proof. Let $K \subseteq \mathbb{E}$ be a paracompac set, then K is S-paracompact and $\mathcal{J}^{-1}(K)$ is S-paracompact in $\mathbb{M}$. By Remark (2.4), $\mathcal{J}^{-1}(K)$ is β-paracompact. Since $\mathcal{L}$ is β-paracompact, then $\mathcal{L}^{-1}\left(\mathcal{J}^{-1}(K)\right) = (\mathcal{J} o \mathcal{L})^{-1}(K)$ is β-paracompact in $\mathbb{W}$. But, $\mathbb{W}$ is an e.d. submaximal space, then $(\mathcal{J} o \mathcal{L})^{-1}(K)$ is paracompact set in $\mathbb{W}$. Thus, $\mathcal{J} o \mathcal{L}$ is paracompact.

Proposition 3.14. Let $\mathcal{L}: \mathbb{W} \to \mathbb{M} = \{y\}$ be a map and $y \notin \mathbb{W}$, then $\mathcal{L}$ is paracompact if and only if $\mathbb{W}$ is paracompact set.

Proof. Since $\{y\}$ is compact, then $\{y\}$ is paracompact, and so, $\mathcal{L}^{-1}(\{y\})=\mathbb{W}$ is paracompact by Proposition 3.9.

Conversely, $\mathbb{M}$ is a singleton point. Consequently, $\mathcal{L}$ is a surjective continuous map and $\mathbb{W} = \mathcal{L}^{-1}(\{y\})$ is paracompact. Thus, $\mathcal{L}$ is paracompact map.

Corollary 3.15. Let $\mathcal{L}: \mathbb{W} \to \mathbb{M} = \{y\}$ be a map and $y \notin \mathbb{W}$, then $\mathcal{L}$ is paracompact if and only if $\mathcal{L}$ is proper.

Proposition 3.16. Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be an injective paracompact map. If $H \subseteq \mathbb{M}$ is open set, then $\mathcal{L}_H: \mathcal{L}^{-1}(H) \to H$ is paracompact.

Proof. Since $\mathcal{L}$ is a paracompact map, then $\mathcal{L}$ is a proper by Proposition (3.4) and by Proposition (2.11), $\mathcal{L}_H$ is proper. Since $\mathcal{L}_H$ is a surjective and continuous map, then by Proposition (3.4), $\mathcal{L}_H$ is a paracompact map.

Proposition 3.17. Let $\mathbb{W}$ be a paracompact space and $\mathbb{M}$ be any space. Then the projection map $\mathbb{Pr}_2: \mathbb{W} \times \mathbb{M} \to \mathbb{M}$ is paracompact.

Proof. Consider the commutative diagram:

$$\begin{tikzcd}
\mathbb{W}\times\mathbb{M} \arrow[rr, "\mathcal{L}\times \mathrm{I}_{\mathbb{M}}"] \arrow[dr, "\mathbb{Pr}_2"'] & & \{p\}\times\mathbb{M} \arrow[dl, "\hbar(\cong)"] \\
& \mathbb{M} &
\end{tikzcd}$$

“Fig. 1”

Notice that $\hbar\colon \{p\}\times\mathbb{M} \to\mathbb{M}$ is a homeomorphism onto $\mathbb{M}$, such that $p \notin \mathbb{W}$ and $\mathbb{Pr}_2\colon \mathbb{W}\times\mathbb{M}\to\mathbb{M}$ is the projection of $\mathbb{W}\times\mathbb{M}$ into $\mathbb{M}$. Further, since $\mathbb{W}$ is paracompact, by Proposition (3.4), $\mathcal{L}\colon \mathbb{W}\to\{p\}$ is a paracompact. Where $\mathrm{I}_{\mathbb{M}}\colon \mathbb{M}\to\mathbb{M}$, then $\mathcal{L}\times\mathrm{I}_{\mathbb{M}}$ is a paracompact by Corollary (3.12), so, by Proposition (3.4) (part ii, $\hbar o\,(\mathcal{L}\times\mathrm{I}_{\mathbb{M}})$ is paracompact. Since $\mathbb{Pr}_2 = \hbar o\,(\mathcal{L}\times\mathrm{I}_{\mathbb{M}})$, then $\mathbb{Pr}_2$ is Paracompact.

Proposition 3.18. Let $\mathcal{L}\colon \mathbb{W}\to\mathbb{M}$ be a surjective continuous map and let $\mathcal{J}\colon \mathbb{M}\to\mathbb{E}$ be an injective, open, and proper map such that $\mathcal{J}o\mathcal{L}\colon \mathbb{W}\to\mathbb{E}$ is paracompact. If $\mathbb{M}$ is Hauosdorff space, then $\mathcal{L}$ is paracompact map.

Proof. Consider the commutative diagram:

$$\begin{tikzcd}
\mathbb{W} \arrow[r, "\mathcal{S}"] \arrow[d, "\mathcal{L}"'] & \mathbb{W}\times\mathbb{M} \arrow[d, "\mathcal{J}o\mathcal{L}\times I_{\mathbb{E}}"] \\
\mathbb{M} \arrow[r, "\mathcal{K}"'] & \mathbb{E}\times\mathbb{M}
\end{tikzcd}$$

“ Fig. 2”

Notice that $\mathcal{S}(\mathrm{x}) = (\mathrm{x}, \mathcal{L}(\mathrm{x}))$ and $\mathcal{K}(\mathrm{y}) = (\mathcal{J}(\mathrm{y}), \mathrm{y})$. By [7, §4, Corollary 2.P.45X], $\mathcal{S}$ is a homeomorphism of $\mathbb{W}$ onto the graph $\mathcal{S}(\mathbb{W})$ of $\mathcal{L}$. Since $\mathbb{M}$ is Hauosdorff space, then by [7, §8, Corollary 2.P.76X], the graph $\mathcal{S}(\mathbb{W})$ of $\mathcal{L}$ is closed set in $\mathbb{W}\times\mathbb{M}$. Since $\mathcal{L}$ is a surjective, then $\mathcal{S}$ is surjective and so, $\mathcal{S}$ is a homeomorphism of $\mathbb{W}$ onto $\mathbb{W}\times\mathbb{M}$ which implies to $\mathcal{S}$ is continuous. Proposition (3. 4) infers that $\mathcal{S}$ is paracompact. Now, since $(\mathcal{J}o\mathcal{L})\times\mathrm{I}_{\mathbb{E}}$ is a paracompact, then $((\mathcal{J}o\mathcal{L})\times\mathrm{I}_{\mathbb{E}})o\mathcal{S}$ is paracompact. But, $((\mathcal{J}o\mathcal{L})\times\mathrm{I}_{\mathbb{E}})o\mathcal{S} = \mathcal{K}o\,\mathcal{L}$, so $\mathcal{K}o\,\mathcal{L}$ is paracompact. Since $\mathcal{J}$ is an injective open proper, then $\mathcal{K}$ is an injective open proper. Therefore, $\mathcal{L}$ is paracompact by Proposition (2.12).

2. Conclusions and future studies

The result of this paper initiated two new types of maps, namely, paracompact map and to investigate his properties under certain conditions. We also found that there are clear relationships between paracompact map and important maps such as closed map and proper map. We will study the new paracompact map in fuzzy space and other spaces.

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