Examining laminar, non-stationary viscoelastic fluid flow between two parallel planes

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Abstract. The generalized Maxwell model is used to handle problems involving the unsteady flow of a viscoelastic fluid in a flat channel under the effect of a constant pressure gradient. Formulas for fluid flow, velocity distribution, and other hydrodynamic characteristics were found. Transient processes during unsteady flow of a viscoelastic fluid in a flat channel are investigated based on the discovered formulas. The analysis's conclusions demonstrated that, at small Debord number values, the procedures of changing a viscoelastic fluid's properties from an unstable to a stationary state essentially don't differ from those of a Newtonian fluid. Exceeding the Debord number relatively unity, it has been established that the process of transition of a viscoelastic fluid from an unsteady state to a stationary state is of a wave nature, in contrast to the transition process of a Newtonian fluid, and the transition time is several times longer than that of a Newtonian fluid. It was also discovered that perturbed processes can arise during the transition. These disturbances occurring in unsteady flows of a viscoelastic fluid can be stabilized by mixing the Newtonian fluid within it. The implementation of this property is important in preventing technical failures or malfunctions.

1 Introduction

In many technical operations, the study of the combined unsteady flow of viscous and viscoelastic fluid in a flat channel under the effect of a constant pressure gradient is employed. It is well known that non-Newtonian fluids' viscoelastic qualities have a major impact on the flow's hydrodynamic parameters. For instance, creating a practical technique to lower the hydrodynamic resistance of flows is crucial when moving heavy, viscous oils and petroleum products across extended distances \cite{1,2}. Especially, during the starting and stopping mode of the working bodies (mechanisms), an unsteady flow is observed. In such cases, the change in fluid flow rate and other hydrodynamic characteristics of the flow differ significantly from the characteristics of conventional Newtonian fluid flows. However, such flows, in addition to the above-mentioned industries, are often used in various technological processes, in chemical technology, in biological mechanics and in acoustics \cite{3,4}.

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Unsteady flows of Newtonian and non-Newtonian fluids, in particular viscoelastic fluids in pipes and channels, are the subject of several investigations [5–7]. I.S. Gromek’s research [5,6] were the first to study the unsteady flow of a viscous incompressible fluid in a cylindrical conduit. He calculated the shear stress, fluid flow, and speed changes on the wall using them. The establishment of hydrodynamic quantities during the flow of a viscous fluid in a cylindrical conduit can be timed using these formulas. Then, this issue was presented as I.S. Gromek’s dilemma in numerous textbooks [5].


In [8], unsteady pulsating flows of a viscous fluid under the influence of a harmonically changing pressure gradient in a circular, cylindrical pipe of indefinite length were examined. Formulas for calculating the distribution of fluid velocity and flow rate were obtained by solving the problem. The speed, flow rate, and other hydrodynamic parameters from the zero initial state in a pulsating flow are established slowly, compared to high oscillation frequencies, and are similar to the parameters of a non-pulsating flow, according to numerical calculations. This is demonstrated at lower values of the dimensionless oscillation frequency. When a flow oscillates at high frequencies, these properties are determined practically immediately.

The problem of unsteady oscillatory flow of a viscoelastic fluid in a flat channel is investigated, based on the Maxwell model [12]. We derive formulas to determine the frequency and dynamic properties. Using numerical experiments, the influence of vibration frequency and relaxation properties of the liquid on the shear stress on the wall was studied. It is shown that the viscoelastic properties of the fluid, as well as its acceleration, are the limiting factors for the use of the quasi-stationary approach. In [13,14], the problems of unsteady flow of viscoelastic fluid in long pipes of annular cross-section were solved. Constant rheological quantities of a viscoelastic fluid were considered to be independent of the rate of deformation; with this condition, the problem is reduced to a solution by a linear differential equation and its analytical solution is obtained using the Laplace transform. The calculation results were carried out for a fluid with constant properties corresponding to a Newtonian fluid, where the development of velocity profiles is of a diffusion nature. The velocity and shear stresses increase monotonically to their stationary values. The elasticity of a liquid gives a wave character to the development of its flow.

In [15], laminar oscillatory flows of viscoelastic fluids, Maxwell and Oldroyd-B, were investigated, where a number of intriguing characteristics that are missing from Newtonian fluid flows are shown. The electrokinetic movement of viscoelastic liquids in a level channel, influenced by a fluctuating pressure gradient, was examined in reference [16]. The fluid is thought to move in a linear mode since it is presumed to be laminar and unidirectional. Since it is assumed that the surface potentials are modest, the Poisson-Boltzmann equation is linearized. Resonant behavior emerges during flow when the Maxwell fluid's elastic characteristics are predominant. Both the efficiency of converting electrokinetic energy and the electrokinetic effects are amplified by the resonance phenomenon.

The above-mentioned works primarily study the oscillatory and unsteady flow of a Newtonian fluid and determine the fluid velocity field under various forms of pressure gradient change. Relatively little research has been done on the variations in maximum longitudinal velocity, fluid flow rate, and tangential shear stress on the wall that occur when viscous and viscoelastic fluid move in an unstable flow. Fluids were typically swapped out for a series of flows with a quasi-stationary distribution of hydrodynamic quantities in hydrodynamic models of unsteady flows. Currently, there is essentially no consensus on the validity of utilizing quasi-stationary features to ascertain the tangential stress field in unstable viscous and viscoelastic fluid flows. Of course, in such
An extended two-fluid Maxwell model is presented in this study to describe the combined flow of viscous and viscoelastic fluids. The study examines the unstable coupled flow of a viscous and viscoelastic fluid in a level channel, influenced by a continuous pressure gradient, using the suggested model. The distribution of fluid flow and longitudinal velocity are calculated using formulae. The analysis focuses on transient events that occur during the initial phase of viscous and viscoelastic flow.

\section{Problem statement and solution method}

We shall examine the solution to the problem of the unsteady flow of a viscous and viscoelastic incompressible fluid between two stationary parallel planes extending in both directions to infinity based on Maxwell's two-fluid model. Let's use \( 2h \) to represent the separation between the walls. In the center of the channel, parallel to the flow, is the horizontal axis \( ox \). The direction of the axis is perpendicular to the axis. A viscous and viscoelastic incompressible fluid's differential equation of motion takes the following form [8–10].

\begin{equation}
\begin{cases}
\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} - \frac{\partial \tau}{\partial y}, \quad \tau = \alpha_1 \tau_1^0 + \alpha_2 \tau_2^0, \\
\tau_1^0 = -\eta_1^0 \frac{\partial u}{\partial y}, \quad \lambda \frac{\partial \tau_2^0}{\partial t} + \tau_2^0 = -\eta_2^0 \frac{\partial u}{\partial y}.
\end{cases}
\end{equation}

Here, \( u \) -longitudinal speed of the mixture; \( p \) -mixture pressure; \( \rho \) -density of the mixture; \( \tau \) - tangential stress of the mixture; \( t \) -time; \( \tau_1^0 \) - is the true shear stress of the Newtonian fluid; \( \tau_2^0 \) -is the true shear stress of the Maxwellian fluid.

The coefficients of dynamic and kinematic viscosity of the mixture can be similarly expressed:

\[ \eta = \alpha_1 \eta_1^0 + \alpha_2 \eta_2^0, \quad \nu = \alpha_1 \nu_1^0 + \alpha_2 \nu_2^0. \]

Here, \( \eta_1^0, \nu_1^0 \) - true dynamic and kinematic viscosities of the Newtonian fluid; \( \eta_2^0, \nu_2^0 \) - true dynamic and kinematic viscosities of the Maxwellian fluid.

Substituting the second, third and fourth equations into the first equation of system (1) to determine the fluid velocity, we obtain the equation

\begin{equation}
\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left( \alpha_1 \eta_1^0 \frac{\partial u}{\partial y} + \alpha_2 \eta_2^0 \frac{1}{1 + \lambda \partial y} \frac{\partial u}{\partial y} \right).
\end{equation}

To solve equations (2), it is necessary to formulate the initial and boundary conditions. We assume that until moment \( t = 0 \), the liquid is at rest. From the moment \( t = 0 \) the liquid moves due to a positive constant pressure gradient \( \frac{\partial p}{\partial x} = const. \) In this case, the initial condition and the no-slip condition on the wall will have the form:

\[ u = 0 \text{ at } t = 0, \quad u = 0 \text{ at } y = h, \quad u = 0 \text{ at } y = -h \]

Carrying out the Laplace-Carson transformation, i.e. passing from the original to the image in equation (2) using the initial and boundary conditions (3), we obtain:
Here, \( \eta * (s) = \left( \frac{\alpha_1 \eta^0}{\eta} + \frac{\alpha_2 \eta^0}{\eta} \frac{1}{1 + s \lambda} \right) = (X + Z \frac{1}{1 + s \lambda}). \)

Using boundary conditions (3), we obtain a solution to equation (4) to determine the speed in the image:

\[
\hat{u}(y, s) = \frac{1}{\rho s} \left( - \frac{d \hat{p}(x)}{dx} \right) \left( 1 - \frac{\cos \left( i \frac{\rho s}{\eta \sqrt{1 - (s \lambda)^2}} \right)}{\cos \left( i \frac{\rho s}{\eta \sqrt{1 - (s \lambda)^2}} \right)} \right).
\] (5)

Using the inversion formula for the Laplace-Carson transform to determine the fluid speed we obtain the following integral expression:

\[
u(y, t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{st} \frac{1}{\rho s} \left( - \frac{d p(x)}{dx} \right) \left( 1 - \frac{\cos \left( i \frac{\rho s}{\eta \sqrt{1 - (s \lambda)^2}} \right)}{\cos \left( i \frac{\rho s}{\eta \sqrt{1 - (s \lambda)^2}} \right)} \right) ds \]. (6)

To calculate the integral (6) over a complex variable, it is necessary to establish residues under the integral expression. Equating the denominator to zero, and taking into account that the roots of the cosine are real numbers, we find:

\[
s = 0 \ \text{and} \ s = -\nu \frac{s_{1,2,n}}{h^2}. \] (7)

Here, \( s_{1,2,n} \) - there are solutions to the transcendental equation

\[
\cos \left( i \frac{\rho s}{\eta \sqrt{1 - (s \lambda)^2}} \right) = 0. \] (8)

All poles of equation (8) will be simple, so we can use the decomposition of the meromorphic function into simple fractions in the form:

\[
\frac{F_1(s)}{F_2(s)} = \frac{\left( - \frac{d \hat{p}(x)}{dx} \cos \left( i \frac{\rho s}{\eta \sqrt{1 - (s \lambda)^2}} \right) - \cos \left( i \frac{\rho s}{\eta \sqrt{1 - (s \lambda)^2}} \right) \right) e^{st}}{\rho s^2 \cos \left( i \frac{\rho s}{\eta \sqrt{1 - (s \lambda)^2}} \right)} = \frac{C_0}{s} + \sum_{i=1}^{2} \frac{C_{in}}{s - s_{in}}. \] (9)

To determine the residue \( C_0 \), we must multiply both sides of equality (9) by \( s \) and then direct \( s \) to zero, and for \( C_{in} \) we must multiply (9) by the differences \( - s_{in} \) and direct \( s \) to the value \( s_{in} \). In this case, to determine parameter \( s_{in} \), we obtain a quadratic equation having two roots, which can be real or complex conjugate

\[ De \hat{s}^2 - \hat{s}(1 + XDea_0^2) + a_0^2 = 0. \] (10)

Here, \( a_0 = \frac{2n+1}{2} \pi, De = \frac{2 \nu}{h^2} \) - Deborah number characterizes the relaxation property of a viscoelastic fluid.

Thus, we find a solution to equation (2) in the following form:
\[ u(y, t) = \frac{h^2}{2\eta} \left(-\frac{d\rho(x)}{dx}\right) \left[1 - \frac{y^2}{h^2}\right] + 32\sum_{n=1}^{\infty} \left[\sum_{i=1}^{\infty} \frac{(-1)^n \cos \left(\frac{2n+1}{2} \pi \frac{y}{h}\right) e^{-\frac{\pi^2 n^2 t}{h^2}}}{(2n+1)^3 \pi^3 (1 - 2\delta \xi_{\infty} + \delta \xi_{\infty}^2)^2} \right] \]

(11)

\[ \frac{u(0,t)}{u_0} = 1 + 32\sum_{n=1}^{\infty} \left[\sum_{i=1}^{\infty} \frac{(-1)^n e^{-\frac{\pi^2 n^2 t}{h^2}}}{(2n+1)^3 \pi^3 (1 - 2\delta \xi_{\infty} + \delta \xi_{\infty}^2)^2} \right] \]

(12)

Here, \( u_0 = \frac{h^2}{2\eta} \left(-\frac{d\rho(x)}{dx}\right) \) - maximum speed for stationary flow of Newtonian fluid.

Expression (11) indicates that as \( t \) tends to infinity, the velocity distribution becomes parabolic. Thus, the solution to the problem of stationary fluid motion between parallel walls is obtained from the solution to the problem of unsteady motion when \( t \) goes to infinity.

### 3 Numerical calculations and discussion

Using the obtained formulas (11) and (12), numerical calculations were carried out to establish unsteady processes of hydrodynamic characteristics during unsteady flow of a viscoelastic fluid. For the convenience of comparing the established Newtonian fluid with the established viscoelastic fluid, we will first conduct a study of the Newtonian fluid. The formula for calculating Newtonian fluid is obtained from formula (8) with \( \lambda = 0, \eta^*(s) = 0, \chi = 1, Z = 0 \). In this case, the solution to the transcendental equation has the form:

\[ \cos \left(i \sqrt{\frac{\rho s}{\eta}} \right) = 0, \left(i \sqrt{\frac{\rho s}{\eta}} \right) = \frac{2n+1}{2} \pi, \delta = -\frac{v}{h^2}, \bar{\delta} = \frac{(2n+1)^2}{4} \]

(13)

Using these expressions from formula (12), one can find calculation formulas for studying Newtonian fluid

\[ \frac{u(0,t)}{u_{\text{omax}}} = \frac{\sum_{n=1}^{\infty} \left[\sum_{i=1}^{\infty} \frac{(-1)^n e^{-\frac{\pi^2 n^2 t}{h^2}}}{(2n+1)^3 \pi^3 (1 - 2\delta \xi_{\infty} + \delta \xi_{\infty}^2)^2} \right]}{\sum_{n=1}^{\infty} \left[\sum_{i=1}^{\infty} \frac{(-1)^n e^{-\frac{\pi^2 n^2 t}{h^2}}}{(2n+1)^3 \pi^3 (1 - 2\delta \xi_{\infty} + \delta \xi_{\infty}^2)^2} \right]} \]

(14)

Based on equation (14), numerical calculations were made and Fig. 1 shows graphs of the change in the ratio of the maximum speed of an unsteady flow to the maximum speed of a stationary flow depending on time. It can be seen that the relative maximum speed during an unsteady flow of a Newtonian fluid, depending on time, monotonically increases to a value corresponding to a stationary flow.
Information about the process of transition of a Newtonian fluid from an unsteady flow regime to a stationary one is given in many literary sources [9-12]. But the process of transition of a viscoelastic fluid from an unsteady flow to a stationary one has not been sufficiently studied. Existing works [13,14] are also devoted to unsteady motions of viscoelastic fluid in some annular pipes, the solution of which is based on the finite difference method. Below we present an analysis of the problem solved on the basis of the generalized two-fluid Maxwell model. For this we use formulas (11) and (12). Based on the obtained formulas, the process of transition from a non-stationary state to a stationary state is analyzed. In this case, the solution of the quadratic equation (7) determines the roots $s_{1n,2n}$ included in formulas (11) and (12). It is known that the quadratic equation (10) has two roots: real different, real equal, and complex concomitant

$$s_{1n} = \frac{1+XDea_0^3 + \sqrt{1-2Dea_0^3(X-2)+X^2De^2a_0^4}}{2De}, \quad s_{2n} = \frac{1+XDea_0^3 - \sqrt{1-2Dea_0^3(X-2)+X^2De^2a_0^4}}{2De}.$$

(15)

In order for the roots of a quadratic equation to be valid, it is necessary that the discriminant $1-2Dea_0^3(X-2)+X^2De^2a_0^4$ in formulas (15) must be equal to or greater than zero. Under such conditions, solution (12) does not change. Analysis of formula (12) is provided in Fig. 2.
Fig. 2. Change over time in the ratio of the maximum speed of the mixture to the maximum speed of the stationary profile during unsteady flow of the mixture (when the Debord number $De = 0.1$, and at different values of the concentration of the Newtonian fluid).

Figure 2 shows that non-stationarity processes in the flow of a viscous and viscoelastic fluid in this instance are essentially the same as non-stationarity processes in a Newtonian fluid. In this instance, creating an unsteady flow of a Newtonian fluid might be used in place of the viscoelastic fluid's unstable flow. Now consider the case when the roots of the quadratic equation (10) consist of complex conjugate roots. This case is observed when the discriminant equation (10) is less than zero. In this case, the solution to the quadratic equation is determined as follows

$$s_{1n} = \frac{1 + XDea_0^2 + i\sqrt{2De}a_0^2(2 - X) - X^2De^2a_0^4 - 1}{2De} = a + bi, \quad s_{2n} = \frac{1 + XDea_0^2 - i\sqrt{2De}a_0^2(2 - X) - X^2De^2a_0^4 - 1}{2De} = a - bi.$$

Here, $a = \frac{1 - XDea_0^2}{2De}, b = \frac{\sqrt{2Dea_0^2(2 - X) - X^2De^2a_0^4} - 1}{2De}$.

In this case, solution (12) has the following form;

$$\frac{u(0, t)}{u_0} = 1 + 32 \sum_{n=0}^{\infty} \frac{2(-1)^{n+1}e^{-\frac{X}{k^2}}}{(2n+1)^3} \left(\frac{X}{M_1^2 + N_1^2}\right) (M_3 \cos b_n \frac{v}{h^2} t + N_3 \sin b_n \frac{v}{h^2} t).$$

Here,

$$M_1 = 1 - 2a_n De + a_n^2 De^2 - b_n^2, \quad N_1 = 2a_n b_n - 2b_n De,$$
Using equation (16), we analyze the results of a numerical calculation of the unsteady flow of a viscous and viscoelastic fluid when the solutions to equation (15) consist of complex conjugate roots. Figure 3 shows the change in time in the ratio of the maximum speed to the maximum speed of a stationary profile during an unsteady flow of the mixture (when the Deborah number has values $De = 3$ and at different values of the concentration of the Newtonian fluid) From Figure 3 it is clear that the transition from an unsteady state to a mixture flows stationary, waves appear, in contrast to a Newtonian fluid, and the transition time of a viscoelastic fluid is several times longer than the transition time of a Newtonian fluid. The main reason for the sharp increase in hydrodynamic quantities during the transition of a viscoelastic fluid can be called the inertial force taken into account in the Maxwell model. It is easy to see that if we remove the action of the inertial force, then it is no different from a Newtonian fluid. The effect of inertial force is explained by the presence of the Debord number, that is, relaxation.

![Figure 3](image)

**Fig. 3.** Change over time in the ratio of the maximum speed of the mixture to the maximum speed of the stationary profile during unsteady flow of the mixture (when the Debord number $De=3$, and at different values of the concentration of the Newtonian fluid)

The fact that the increase in the Debord number exceeds the flow rate of the viscoelastic fluid is 4.5–5 times compared to the flow rate of the Newtonian fluid in the transient process. These changes are presented in Fig. 3. From this it can be seen that at higher Debord numbers, the process of transition of a viscoelastic fluid from an unsteady state to a stationary state has a wave form, in contrast to the transition process of a Newtonian fluid, and the transition time is several times longer than the transition time of a Newtonian fluid. It was also discovered that perturbed processes can arise in the transition, which will be stabilized by mixing the Newtonian fluid in it. The implementation of this property is important in technical and technological processes, in preventing technical failures or malfunctions.
4 Conclusion

The problem of unsteady flow of a viscoelastic fluid under the effect of a constant pressure gradient in a flat channel is solved based on the proposed two-fluid Maxwell model. In order to address the issue, the Laplace-Carson transformations were utilized, and formulas for the distribution of the fluid flow and velocity profile in an unstable viscoelastic flow were established. The methods by which the properties of a viscoelastic fluid in a flat channel change from an unstable to a stationary condition are examined using the established formulas. Based on the results of the analysis, it was shown that the transient processes of the unsteady flow of a viscoelastic fluid, at small values of the Debord number, are not fundamentally different from the transient process in a Newtonian fluid. Exceeding the Debord number relatively unity, it has been established that the process of transition of a viscoelastic fluid from an unsteady state to a stationary state has a wave form, in contrast to the transition process of a Newtonian fluid, and the transition time is several times longer than the transition time of a Newtonian fluid. It was also discovered that perturbed processes can arise in the transition, which will be stabilized by mixing the Newtonian fluid in it. The implementation of this property is important in technical and technological processes, in preventing technical failures or malfunctions.

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