Physically based method for determining the Hausdorff's topological dimension of capillary-porous media

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Abstract. The soil structure the solid phase and pore space of the soils are a set of self-similar parts of each other at different levels (for example, on level aggregates, micro-aggregates or elementary particles of soil). Fractal models of the soil structure best describe this spatial composition of soils. For a quantitative description of the soil structure, the dimension Hausdorff (D) is adopted, based on the premise different of scales in the soil structure. The existing methodology for determining the Hausdorff dimension using computational methods represents a series of labor-intensive operations including either special analysis of images or tomograms, or production of analyzes such as dry sieving and soil granulometric analysis. The development of less labor-intensive algorithms for determining the Hausdorff dimension is relevant. The direct, physical method for determining the parameters of the structure of the object of study presented in the work is preferable to the calculation method. The testing of the method proposed in the work consists in comparing the results of determining the values of the Hausdorff dimension obtained by the generally accepted methodology and the method for determining D by moisture filtration. The research results are summarized in a table and show almost complete convergence of the Hausdorff dimension values obtained on the different methodologies basis.

1 Introduction

In the modern realities of fractal analysis of soil structure, the theory of fractal fraction is widely known and applied, which is presented in detail in the works of foreign researchers [1-5]. The Mass fractal model or Pore fractal model are used, respectively [5-7]. Famous models of soil structure (shown in Fig. 1) are the ideal volume fractal – Menger sponge and pore-solid-fractal-model-modification (PSF-M) [8, 9].

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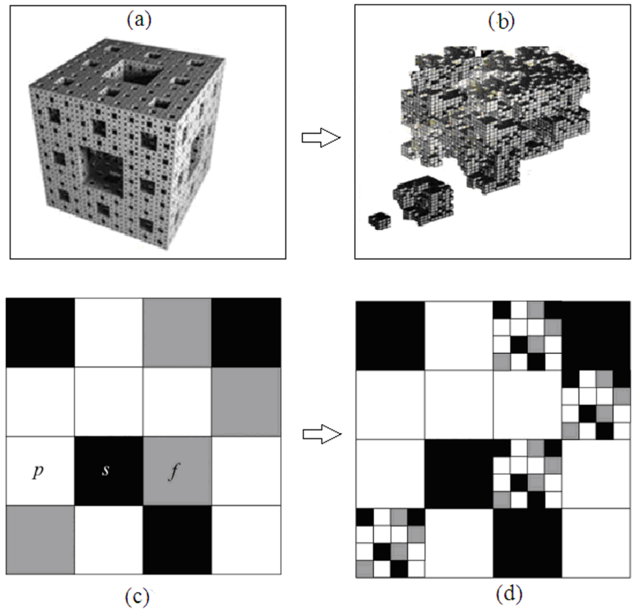

Fig 1. a - Menger sponge; [2; 3], b - Menger sponge soil structure model [6; 9]. c and d - PSF-M model [5-7].

A generalizing, model of soil structure is the pore solid fractal model (PSF), which is a tool providing the physical basis for the parameters of hydrophysical models, including the Van Genuchten model [10-13]. These models are presented in the works of foreign and domestic researchers [2-13].

2 Materials and methods

Common to the ensemble of particles of the solid phase of soils or the ensemble of pores is scale in variance. The total length of pores within an invariant or self-similar structure depends on the scale of identification, therefore in fractal models the main parameter for the quantitative description of the structure of the pore space is the fractional topological dimension the Hausdorff dimension (denoted by D) [10]. The Hausdorff dimension acts as the main fractal parameter and determines the entire geometry of fractals. The Hausdorff dimension is the basis for determining the group of "fractal dimensions" of geometric objects [11]. Fractal dimension depending on the fractal construction method (approximations to the integer dimension of the surface or volume – "top or bottom") there may be more or less integer dimension of the space, which we will further call the Euclidean dimension. In soil physics and soil mechanics, it is customary (not consciously) to use an approximation to the Hausdorff dimension from above in calculations – this approximation is an expression of the concept of "capacity" of the volume of space containing the fractal. Therefore, the fractal topological dimension in modeling the structure in soil physics is always strictly less than the Euclidean dimension of volume equal to 3. A fractal can be defined as a set with the property of multiscale (scaling). However, this is true only for regular, constructed fractals (for example, Menger's sponge). Natural fractals, including soil structures, have probabilistic properties and belong to the group of stochastic fractals. Thus, all fractal dimensions are classified as metric and

probabilistic, respectively, by the method of introduction. For more information about the fractal dimension (Hausdorff), see, for example [5,7, 9].
The formal definition of the Hausdorff dimension is given by Equation 1:

$$D = \lim_{m \to 0} \frac{\log L(m)}{\log\left(\frac{1}{m}\right)} \text{ или } D = \frac{\log L(m)}{-\log m} \quad (1)$$

were, L represents the length or the characteristic linear size of the area or volume of the fractal in image analysis, or the extent of a numerical or time series in the method of calculating the dimension d of linear systems.

The Russian scientist Goncharov V.D., a student of A.M. Globus, studied in detail the possibilities of applying the theory of fractals to the study of soil structure. In particular, V.D. Goncharov widely used the concept of probability. In other words, the fractal structure of the soil pore space has a probabilistic character. In some cases, the probability of a fractal organization of the soil structure is equal to one, in other cases, a fractal structure is unlikely. For the PSF-M model, the Hausdorff dimension of a capillary-porous medium is calculated using the equation proposed by Perrier E, and modified by Goncharov V.D. in 2004, he also proposed formulas for calculating the fractal dimension of the soil structure, taking into account probability. He widely applied the concept of probabilistic fractal dimension. The corresponding formulas can be found in the primary sources [2-8]. Here we point out that such a development of ideas about the fractal dimension of capillary-porous media (soils) has led to an extreme complication of computational algorithms

In the case of incomplete fragmentation, it is necessary to know the number of fragmented regions in the soil and the probability of their fragmentation (i.e., conduct an aggregate analysis of the soil):

$$D_f = \frac{\log(PF)}{\log F} \quad (2)$$

Df - Hausdorff dimension under incomplete soil fragmentation, P - the probability of fragmentation.

The calculation method D based on the known physical parameters of the soil structure is a long and time-consuming operation. Since these physical parameters need to be found experimentally. The development of simpler and less time-consuming methods for determining the fractal dimension of porous media is still an urgent goal and task of soil hydrophysics [11-12]. This work proposes an independent physical method for determining the Hausdorff dimension based on the Hurst effect when studying natural processes with zero mean.

3 Objects and methods

In nature, random processes with a zero mean are common (e.g., Brownian motion, seasonal cycles that do not lead to changes in environmental parameters). Brownian motion is a continuous random process . From a physical perspective means process does not decay. With mathematical perspective, the process is autocorrelated. This implies the absence of correlations in the consecutive increments of the fluctuations of the characteristic parameter value X(t,w) and the constancy of the spectrum across all frequencies (w) $f(w) = \text{const}$. The variance of such a process is proportional to the entire considered period of time. $Sx_t \approx T$ In strict mathematical notation, this can be expressed as: $Sx_t = (\sqrt{T})^2; при\, T \geq 0$ or $Sx_t = t^{2H}; при\, t \geq 0; и\, 0 \leq H \leq 1$

The essence of the Hurst effect is that in certain cases (e.g., during moisture filtration through soil, the occurrence of floods in large rivers, or the study of financial risks on the stock market), the exponent T not equal to $\sqrt{}$ or 0,5, respectively, so the product 2H not equal to [14, 15]. If the exponent 2H<1, then such a random autocorrelated process is fractal, and in the case of moisture filtration through a soil sample, it reflects the fractal distribution of soil elements. The fractal dimension of the soil volume is calculated by the relation D=2+H [16]. This view allows us to develop a method for the direct physical determination of the fractal dimension of the soil pore space by the method of linear systems, according [11; 4]. The graphical representation of the nature of moisture filtration through a soil volume is a time series—the dependence of the volumetric water flow on time. Known method for definitions the fractal dimension of a discrete time series. The total length of the time series changes with the scale of measurement of this length. In logarithmic scale, a plot of the logarithm of the time series length versus the logarithm of the measurement scale is constructed: $\log L(m) = f(\log m)$. The fractality index "H" is determined as the exponent of the approximating function of the form: $y = AX^H$. By defining H , the dimension D is calculated.

Varieties of arable soils have been selected as objects of research. By classification WRB: Umbric ALBELUVISOLS, sandy soils on abraded moraines and Umbric ALBELUVISOLS, loamy sand soil the village of Menkovo, Gatchinsky Region, Leningrad Area. WRB: Gleyic PODZOLS, medium-loamy, AIC Bugry, Leningrad Region; Umbric ALBELUVISOLS (gleyic) sandy soil, Rodina settlement, Pskov Region PNIISH, southern agrochernozem (By WRB: Calcic CHERNOZEMS), weakly humused, heavy loam on loess, Krasnogvardeisky District, Crimea (field of the "NIISH"). The soil sample were taken along soil horizons a depth to 80 cm. The granulometric composition of the soil was determined according to the method described in GOST 12536-2014 (the end of the analysis is the pipette method). The content of physical clay (particles with an effective diameter of less than 0.01 mm) varied across genetic horizons and soil profiles from 14.5% to 65%.

4 Results and discussions

A peculiar effect is known in large rivers: the arrival of a flood wave, that is, floods on such rivers have several pronounced peaks. Secondary flood peaks often occur at the height of the summer drought some time after the decline of the main flood level. Water is trapped by the soil, some significant volume of water seems to wander through the soil voids throughout the riverbed or part of the riverbed. At some point in time, the spontaneously accumulated moisture is released and enters the river forming a secondary flood peak. The reasons for this phenomenon are debatable. One possible explanation is the fractal structure of soils and soils. Water takes a long time to cross the fractal space. The topological complexity of the fractal soil structure leads to the similarity of the siphon effect. When moisture gradually accumulates in the voids of the soil and when fractally organized pores overflow, the entire volume of water is spontaneously and immediately released, forming a flood situation on the river for the second time. Of course, the hypothesis accepted by the authors requires comprehensive discussion and discussion. Perhaps it is not true in principle. The research conducted by the authors suggests that the soil very often has a fractal structure. This situation can be simulated in a filtration device. If the soil is a fractal, then the filtration of moisture through the soil will not be uniform, especially at the initial stage

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of filtration. When studying the filtration of moisture through the soil according to GOST 25584-2016, it is necessary to plot the dependence of the volumetric flow rate of water Q on time - a time series. There are three possible test outcomes. The pronounced Hurst effect that occurs during filtration in the case of fractal soil structure is shown in Fig. 2.

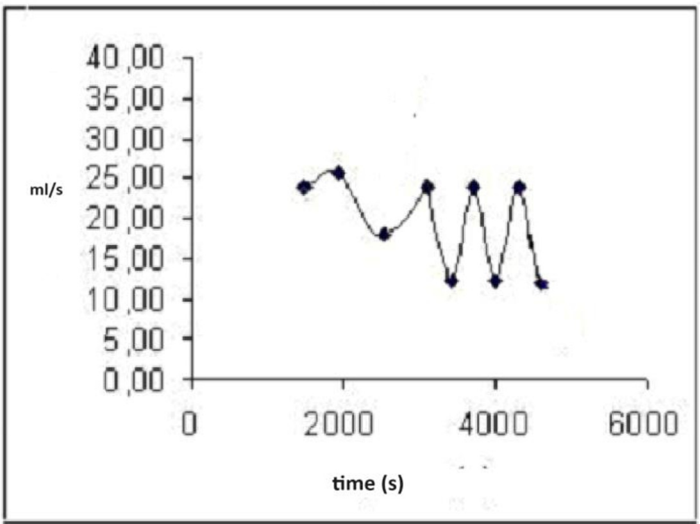

Fig. 2 Dependence of the volumetric flow rate of water Q on time. The presence of a pronounced Hurst effect. According to our latest data. Soil: Umbric ALBELUVISOLS. With signs of gley; sandy granulation.

In addition, other options are possible. Earlier, our research showed [10-12] that in some cases we get an exponent. That is, the filtration of water through the soil sample has a smooth character. This type of filtration is also required by GOST 25584-2016 when conducting soil filtration tests. In this case, the soil structure does not have a fractal structure. Another option is possible when, on some segment of the filtration curve, the filtration does not have a smooth character. The moisture filtration process is autocorrelated only on a small section of the curve. That is, filtration is not stationary, but it also does not have a pronounced autocorrelation, that is, the process fades, gradually passes into the stage of stationary filtration. This filtration process is typical for most soils. On the one hand, the presence of a self-oscillation area (sinusoids) makes it possible to apply the method of linear systems of fractal theory to calculate the fractal dimension of soils, and on the other hand raises some doubts.

Soils are special natural formations, many scientists assume that the fractal structure of soils is mandatory. And other researchers don't quite agree with this. Another question is how accurate our methods of determining fractal dimension are. Undoubtedly, the calculation methods proceed from the unconditional fractal organization of soils and on the basis of this a priori assumption, algorithms for calculating the fractal dimension for the soil space organized as a fractal are built. The hypothesis we put forward and filtration studies in this vein, the methodology being developed, show that soils are more likely to have a fractal structure than not. It is more correct to formulate it this way: soils, especially native ones, predominantly have a fractal organization of the pore space -their structure as a whole. As for the soils in agricultural use, obviously everything depends on the duration of this use. In the process of economic activity, the soil structure loses its fractal structure.

The validation of the method involves comparing the results of determining the dimension D using the method of studying the non-stationary moisture filtration region through a soil sample with the canonical method. The Hausdorff dimension of the soil structure was calculated using the canonical method based on data from soil dry sieving (aggregate analysis), soil solid phase density, bulk density, and granulometric analysis. The research results are presented in Table 1.

Table 1. The Hausdorff's dimension values of research soils

Name	Decrement (H)	Dimension Hausdorff (Hurst effect)	Dimension Hausdorff common method of calculation
Umbric ALBELUVISOLS (gleyic) sandy soil, horizon C. [1]	0,69	2,69	2,71
Umbric ALBELUVISOLS (gleyic) of sandy granulometric composition, horizon C, model (disturbed structure). [10]	0,76	2,76	2,73
Umbric ALBELUVISOLS, typical, loamy sand soil, horizon B. [1]	0,72	2,72	2,74
Umbric ALBELUVISOLS, typical, loamy sand soil, horizon C.	0,54	2,54	2,59
Calcic CHERNOZEMS, heavy loamy soil, horizon A (10-20 cm)	0,83	2,83	2,87
Gleyic PODZOLS, medium-loamy horizon B1. model (fraction less than 2 mm).	0,49	2,49	-
Calcic CHERNOZEMS, heavy loamy soil, horizon B (70-80 cm) [1]	0,88	2,88	2,88

The research results show that there is a systematic but slight underestimation of the D values determined through moisture filtration.

Figure 1 shows a pronounced Hurst effect and the Hausdorff dimension is calculated without restrictions over the entire length of the time series. In this case, the moisture filtration process is completely autocorrelated, that is, it does not fade in time. The soil structure has a fractal structure. When the test result is obtained (see our previously published works), in which moisture filtration is generally attenuated, tending to the usual exponential, but there is an area of a non-attenuating process, then in this area the Hausdorff dimension should be calculated by the method of linear systems of fractal theory. Alternatively, to calculate the fractal dimension of the time series, R/S analysis [14] is applied along the entire length of the series, and not only in the selected area. In the case when, at the initial stage of soil filtration tests, the test result is a canonical exponential curve of the field of stationary moisture filtration according to the requirements of GOST 25584-2016 for testing the filtration properties of soils and soils. Here, the initial area of turbulent moisture filtration, due to the fractal organization of the soil structure proposed here, is missing. In this case, the authors propose to use existing regular models of soil structure (for example, the balls model [13]) to model the pore space of soils and hydrophysical functions. The proposed approach allows you to choose (at least) the type of soil structure model (regular or fractal) for a particular soil. With such a geometric model, we can simulate the structure of a capillary - porous body and calculate the hydrophysical parameters of soils using dynamic or physical modeling methods, without resorting to direct experimental determination of parameters.

It should be said that the theory of fractals can be successfully applied in the development of some special methods of soil analysis [17] and especially phenomena arising in the processes of soil colloid transfer or the description of phenomena of partial hydrophilicity of soils. The phenomenon of uneven distribution of colloids in the volume of soil space — the fractal distribution of colloids, in turn, may be one of the causes of the hysteresis of some hydrophysical functions of soils.

Equally important, for example, is that native soils are more likely to have a fractal structure, while arable soils, disturbed soils and structural soils (especially the latter) are more likely to have no fractal structure. In addition, the degree of "fractality", that is, which fractals regular or irregular the soil consists of, may reflect the degree of soil degradation. Various manifestations of soil degradation are a complex phenomenon, a set of processes leading to a change in the functions of the soil as an element of the natural environment, a quantitative and qualitative deterioration of its composition and properties, a decrease in the natural and economic importance of land, loss of soil fertility, all these manifestations of degradation are determined by the state of the soil structure at a given time. In this case, the fractal dimension is a quantitative indicator of the development of the soil degradation process. The more soil structure disorders there are, for example, loss of soil aggregate state, the less likely its fractal structure is. At the first stage of its degradation, the soil will move from the state of a regular fractal to the state of a probabilistic fractal, and then a large-scale series of different fractals may form and the soil will acquire a multifractal structure. Complete soil degradation will correspond to the absence of fractal organization of the soil structure.

5 Conclusions

- Our methodology for determining the Hausdorff dimension, tested by calculating the fractal dimension using well-known methods, shows good convergence of the values of the fractal dimension of the soil structure.
- There is a systematic, slight underestimation of D values determined by moisture filtration.
- The method proposed here for determining the Hausdorff dimension of the soil structure, in the case of the Hurst effect when moisture is filtered through the soil, is completely justified.
- The development of a physically sound method for determining fractal dimension requires further research, including taking into account the phenomenon of hysteresis [18-23] of the dependence of moisture potential on soil moisture content.
- Our methodology for determining the Hausdorff dimension reopens the question of the fractal organization of soils. Is this a universal phenomenon or not. The authors tend to adhere to the former, propose to develop a discussion, urge to discuss the methodology

itself, but at the stage of research presented here, the method may well be used to determine the existence of a fractal structure of the soil structure.

- In our opinion, several important aspects of the application of fractal theory to soils are touched upon in the work.

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