

# Maximum entropy production principle in forest dynamics modelling

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**Abstract.** Forest ecosystems are vivid representatives of open non-equilibrium systems. The existence of extreme principles in "ecological thermodynamics" is a subject of discussion in the works of many physicists, ecologists and researchers dealing with non-equilibrium thermodynamics. At the same time, the problem of the connection between the principles of maximum and minimum entropy production has been studied in detail enough in the works of L. M. Martyushev et al. However, for forest ecosystems, the works that point out the connection of these fundamental principles are clearly insufficient. Usually, these principles are opposed to each other. In the proposed work, within the framework of a unified approach, the dependencies of the entropy production density and the entropy density have modeled using the example of a pine (*Pinus sylvestris*) stand of the 1-grade forest site capacity. It has shown that entropy production and entropy production density take both maximum and minimum values in the process of ecosystem evolution.

## 1 Introduction

In forest ecosystems modelling, silvicultural constants play a very significant role, in particular the time of reaching the maximum value of the forest stand biomass [1], the value of the biological maturity age [2], and the base age in the dynamic growth model [3]. Usually the use of such constants increases the accuracy of modelling. Although, as it is known [4], the modelling approximates reality, but does not duplicate it or reproduce it exactly, because it is impossible due to the high complexity of the modelled object (in this case, we are talking about forest ecosystems).

The application of such constants in modelling is of particular importance in calculations of an important thermodynamic characteristic of open non-equilibrium systems, which undoubtedly include forest stands: entropy density and entropy production density [5] (hereinafter, for short, entropy and entropy production). At present, the question of applying the second law of thermodynamics to biological and ecological problems is still debated [6], and both issues related to the definition of the entropy concept itself and to the formulation of extreme principles of entropy production are touched upon. It seems to us that these problems are generally solved in the works of L.M. Martyushev [7, 8]. We consider

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especially important the question of the connection between the Maximum entropy production principle (*Maximum entropy production principle* MEPP) and Prigogin's principle of minimum entropy production [9, 10]. According to the conclusions in the works of L.M. Martyushev et al. (for example, see [7, 8]), these are absolutely different variational principles, in which although the extremum of the same function (entropy production) is sought, but different constraints and different variation parameters are used. These principles need not be contrasted, since they are applicable to different stages of the evolution of a nonequilibrium system. A hierarchy of processes in the ecosystem is observed. On small timescales, the system maximises entropy production at given fixed generalised forces at the considered time moment, and as a result the MERR principle is valid. On large time scales, the system starts to vary free generalised forces and entropy production decreases, reaching a minimum value in accordance with Prigogin's principle [9, 10].

In our works [11, 12], based on the silvicultural constant (the time of reaching the maximum value of the plantation biomass  $t_{max}$ ) it has obtained that entropy production for forests reaches the minimum positive value in accordance with Prigogin's principle, and the value of entropy at this point in time has obtained. In the present work, using the MERR principle and an important constant from our point of view – the age of biological maturity  $t_{inf}$  a model of calculation of entropy density and entropy production density for pine (*Pinus sylvestris*) stands of 1-grade in a wide interval of time was developed in a wide range of ages from 10 to 300 years. The starting point of the proposed model is the method of calculation of stand growth dynamics based on the thermodynamic approach [1], which we developed earlier and improved in [2, 12].

## 2 Materials and methods

The entropy balance,  $dS$  of an open ecosystem can be represented as

$$dS = d_i S + d_e S \quad (1)$$

where  $d_i S$  is the entropy change caused by internal processes;  $d_e S$  is the entropy change caused by external exchanges. While  $d_i S$  is always positive, as dictated by the second law, the second term of the equation  $d_e S$  can be negative and numerically greater than  $d_i S$ . Allowing the resulting entropy balance of the system to be negative as well. According to (1), the rate of entropy change will take the form

$$\frac{dS}{dt} = \frac{d_e S}{dt} + \frac{d_i S}{dt} \quad (2)$$

Let us further consider a stand of trees with biomass  $M$ . Theoretically, we should expect that the main contribution to entropy change comes from two processes: total biomass growth and cell division. To separate these effects, we consider the change in specific entropy (entropy density), that is, the entropy per unit of biomass  $\sigma = S/M$ :

$$dS/dt = M d\sigma/dt + \sigma dM/dt \quad (3)$$

The process of differentiation leads to a decrease in specific entropy, as the order in the system increases, and biomass growth corresponds to the positivity of the  $dM/dt$  derivative, i.e., the change in ecosystem entropy is determined by the combination of negative (differentiation) and positive (growth) derivatives. From equations (2) and (3) we have

$$\frac{dS}{dt} = \frac{d_i S}{dt} - \left| \frac{d_e S}{dt} \right| = M \frac{d\sigma}{dt} + \sigma \frac{dM}{dt} \quad (4)$$

The minus sign corresponds to an increase in the orderliness of the system until the stationary state is reached. For this purpose, we present the expression for entropy in the form

$S = \sigma M = \rho \sigma V$ , where  $\rho$  is the biomass density, and  $V$  is the stand volume. Following [4, 11], suppose that

$$\frac{d_i S}{dt} = \alpha(t)M, \quad \frac{d_e S}{dt} = \beta(t)F \quad (5)$$

where  $\alpha$  and  $\beta$  are time functions subject to the formalised modelling process, and  $F$  is the surface area of the stand. Note that the functions  $\alpha$  and  $\beta$  have a similar meaning as the corresponding terms in the definition of entropy change with time  $t$  in a local unit volume [9, 10]:

$$\frac{ds}{dt} = \sigma - \text{div} \mathbf{J}_s \quad (6)$$

$$\sigma = \sum_i X_i J_i, \quad (7)$$

where  $s$  is entropy density;  $\sigma$  is entropy production density (On the basis of the second beginning of thermodynamics, entropy production is always greater than or equal to zero. In our notation,  $\sigma$  is the entropy density, not the entropy production density.);  $\mathbf{J}_s$  is entropy flux density vector, additively depending on generalised thermodynamic flux densities  $J_i$ ;  $X_i$  are generalised thermodynamic forces (depending on the problem conditions index  $i$  is used to denote both different fluxes and vector components).

Using (4) taking into account relations (5) of entropy production density, we have

$$\frac{d\sigma}{dt} + \frac{\sigma}{M} \frac{dM}{dt} = \alpha(t) - \beta(t) \frac{F}{M} \quad (8)$$

We have obtained an inhomogeneous first order differential equation whose general solution is well known [13]. For the equation of the form

$$\frac{dy}{dx} + f(x)y = g(x), \quad (9)$$

if the functions  $f(x)$  and  $g(x)$  are continuous on the segment  $a < x < b$ , the integral curve passing through the point  $(\zeta, \eta)$  is defined by equation

$$y = e^{-F}(\eta + \int_{\zeta}^x g(x)e^F dx), \quad (10)$$

where  $F(x) = \int_{\zeta}^x f(x)dx$ .

For our case we obtain

$$f(t) = \frac{1}{M(t)} \frac{dM(t)}{dt}, g(t) = \alpha(t) - \beta(t) \frac{F(t)}{M(t)} \quad (11)$$

For the initial value of time  $t = 0$ , entropy is zero, since the stand as an ecosystem does not yet exist, i.e.  $\sigma = 0$ . Under these initial conditions, the solution of equation (11) is as follows

$$\sigma(t) = \frac{1}{M(t)} \int_0^t [\alpha(t)M(t) - \beta(t)F(t)]dt \quad (12)$$

Then we apply formalised modelling to determine the explicit form of the functions  $\alpha(t)$  and  $\beta(t)$ , and then, according to the principles of system analysis, we conduct a computational experiment [14].

### 3 Results and discussion

The choice of the explicit form of dependencies  $\alpha(t)$  and  $\beta(t)$  will be made using the behaviour of the functions at the beginning of the stand life cycle, i.e. when the stand reaches the age of physiological maturity, and when approaching the time corresponding to the beginning of

the stationary regime, which we denoted as  $t_{max}$ . The age of physiological maturity is an important constant in silvicultural practice and was used in our modelling of stand growth dynamics [2]. From a mathematical point of view, at this time point the biomass of the stand as a function of time passes the inflection point  $t_{per}$ , where the second derivative of the function  $M(t)$  is zero.

We assume that entropy production reaches its maximum value at the age of physiological maturity. Based on this assumption, we choose the following dependence for the function  $\alpha(t)$ :

$$\alpha(t) = A \exp\left(-\frac{(t-t_{max})^2}{2d^2}\right) \quad (13)$$

In accordance with Prigozhin's principle, as it follows from works [11, 12], the function  $\alpha(t)$  at  $t = t_{max}$  reaches the minimum positive value. And at this time point, the value of entropy is known from the calculations of works [11, 12]. While the value of the function  $\beta(t)$  is also known at this point from the same works. This information allows us to determine the constant  $A$  out of the known value of entropy at the point  $t_{max}$ . The parameter  $d$  of the model is determined during the computational experiment for large values of time, when the entropy should increase after reaching the minimum positive value.

The choice of an explicit expression for the function  $\beta(t)$  is also based on the calculation results of [11, 12]. In these works, the function  $\beta(t)$  is considered to be a constant value. From the point of view of the occurring during stand growth physics processes, this function is responsible for high-quality energy flows into the system and low-quality energy flows out of the ecosystem. The definition of the magnitude of these fluxes and their time dependence can be found in [15]. It is known that they are largely determined by the albedo of the stands reflecting surface. At the growth beginning, the function  $\beta(t)$  increases reaching its maximum value at  $t = t_{max}$ , and then it slowly decreases tending to zero at  $t \rightarrow \infty$ . Based on these considerations, we choose the following dependence for the computational experiment

$$\beta(t) = \frac{Q}{1+(t-t_{max})^2} \quad (14)$$

where the model parameter  $Q$  is found from the calculations of [11, 12].

The stand surface area and biomass are related by the relationship

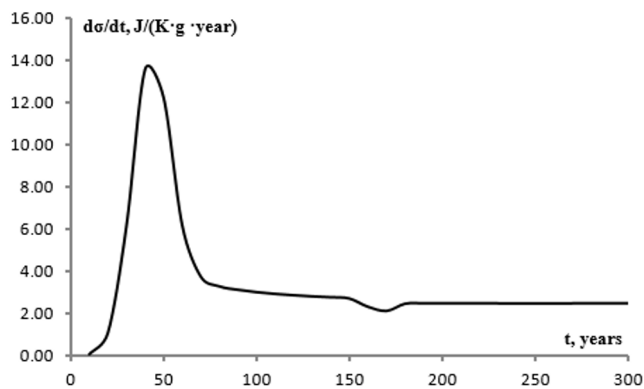
$$F(t) = gM^q \quad (15)$$

where  $g$  and  $q$  are parameters of the ecological-physiological model (EPM) [1].

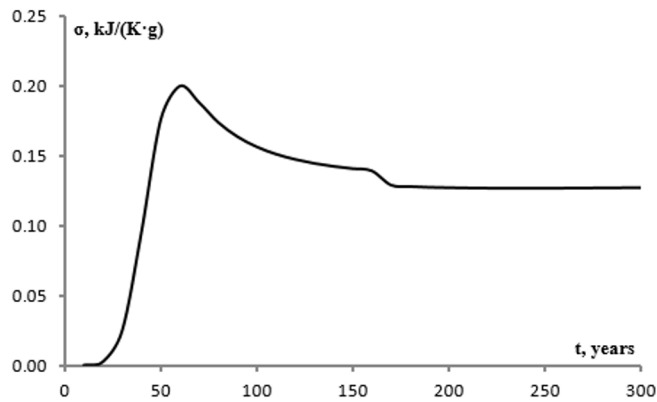
To conduct the computational experiment, a program was created in the RTC Mathcad Prime 4.0 environment, and it can be implemented in earlier versions of Mathcad as well.

The results of this computational experiment are shown in Figures 1 and 2, as well as in the table 1.

The analysis of the results of the computational experiment gives next. The curve of the entropy production density dependence reproduces both the maximum value  $d\sigma/dt$ , and the minimum value of the entropy production density (Fig. 1). And in the vicinity of the inflection point  $t_{inf}$ , the maximum is pronounced, that indicates a good agreement between the model and our initial assumption. Modelling with an increased parameter  $d$  leads to both a smoothing of the maximum and a shift towards longer times. Therefore, we give the value of the kink parameter  $d = 10$  as the optimal value for the model.



**Fig. 1.** Results of calculating entropy production density as a function of time for a pine (*Pinus sylvestris*) stand of the 1-grade forest site capacity.



**Fig. 2.** Results of entropy density calculation as a function of time for a pine (*Pinus sylvestris*) stand of the 1-grade forest site capacity.

**Table 1.** Results of entropy density and entropy production density as a function of time for a pine (*Pinus sylvestris*) stand of the 1-grade forest site capacity.

| Time <i>t</i> , years | Entropy density $\sigma$ , kJ/(K·g) | Entropy production density $d\sigma/dt$ , J/(K·g·year) |
|-----------------------|-------------------------------------|--------------------------------------------------------|
| 10                    | 0.0000800851                        | 0.0390701                                              |
| 20                    | 0.0026073192                        | 1.0390016                                              |
| 30                    | 0.0248712329                        | 6.1136828                                              |
| 40                    | 0.0952092101                        | 13.5407914                                             |
| 50                    | 0.1752699286                        | 12.2144369                                             |
| 60                    | 0.1994966135                        | 6.1628916                                              |
| 70                    | 0.1877383737                        | 3.7209608                                              |
| 80                    | 0.1734974704                        | 3.2602630                                              |
| 90                    | 0.1632711536                        | 3.0966491                                              |
| 100                   | 0.1560667186                        | 2.9843590                                              |
| 110                   | 0.1508749796                        | 2.9007232                                              |
| 120                   | 0.1470668003                        | 2.8368385                                              |
| 130                   | 0.1442368074                        | 2.7861092                                              |
| 140                   | 0.1421074139                        | 2.7413160                                              |
| 150                   | 0.1404589701                        | 2.6822063                                              |
| 160                   | 0.1388568000                        | 2.2908198                                              |

|     |              |           |
|-----|--------------|-----------|
| 170 | 0.1286793876 | 2.0936197 |
| 180 | 0.1276614370 | 2.4357906 |
| 190 | 0.1271652468 | 2.4558381 |
| 200 | 0.1268492517 | 2.4573416 |
| 210 | 0.1266497182 | 2.4560198 |
| 220 | 0.1265344253 | 2.4545146 |
| 230 | 0.1264818739 | 2.4534144 |
| 240 | 0.1264762933 | 2.4528165 |
| 250 | 0.1265056644 | 2.4526803 |
| 260 | 0.1265606676 | 2.4529286 |
| 270 | 0.1266340194 | 2.4534807 |
| 280 | 0.1267200125 | 2.4542631 |
| 290 | 0.1268141783 | 2.4552130 |
| 300 | 0.126913031  | 2.456278  |

The minimum value of  $d\sigma/dt$  is observed at time  $t = 240$  years. Certainly it is larger than that obtained in the works [11, 12]. However, as it was emphasized in [1], the stationary regime is a process and its beginning marks the initial value of the minimum of entropy production. With increasing time, the value of the function  $d\sigma/dt$  differs little from the minimum value (see table 1), that is observed in the model experiment.

In the vicinity of the point  $t = t_{max}$ , we observe a violation of the monotonic character of the function  $d\sigma/dt$ . In our opinion, it is due to the suspicion of a special point of the function at that point in time. When shifting the point  $t = t_{max}$  by 5 years, this dip practically disappears, as it is shown in the graph (Fig. 1).

The dependence of entropy on time is in general adequate to our assumptions (Fig. 2), although at the "special" point  $t = t_{max}$ , a kink in the curve is observed. The maximum value of entropy density relative to the point  $t = t_{inf}$  is observed with a shift of 20 years compared to the maximum of entropy production density, i.e. entropy grows, but its growth rate slows down.

In general, the dependence of entropy on time is adequate to our assumptions (Fig. 2), although a kink in the curve is observed at the "special" point  $t = t_{max}$ . The maximum value of entropy density is observed with a shift of 20 years relative to the point  $t = t_{inf}$  compared to the maximum of entropy production density. I.e. entropy grows, but its growth rate slows down.

4 Conclusion

1. The connection between the maximum entropy production principle (MEPP) and Prigozhin's minimum entropy production principle is well proven in a computational experiment using the developed formalised model. Both maximum and minimum values of entropy production density are obtained for different time periods of ecosystem evolution within one approach.
2. Our approach is based on the principles of ecological-physiological model (EPM) construction [1] taking into account the improvement and modification proposed in [11,12], where silvicultural constants had applied.
3. Determination of parameters for the dependence of functions  $\alpha(t)$  and  $\beta(t)$  with application of physically justified values of entropy density and entropy production density allowed to obtain quite justified their values within one approach for a pine (*Pinus sylvestris*) stand of the 1-grade forest site capacity.
4. The method proposed in this paper can be applied to stands of other species and growing conditions, for which it is known the dynamics of biomass change.

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