Using fractal analysis in short-term river flow forecasting

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Abstract. This article discusses an approach that improves the accuracy of short-term river runoff forecasting through the use of fractal diagnostics and fractional differential equations. A method for calculating the fractal dimensions of time series is considered, based on a correlation integral, allowing for the selection of the optimal number of phase variables for modeling hydrological processes. It is shown that the use of fractional differential equations corresponds to the actual structure of time series and improves forecast quality, particularly during the rise and fall of spring floods and autumn floods. A comparative analysis of the effectiveness of forecast models is provided. It is confirmed that the proposed approach outperforms traditional methods, increasing the reliability of forecasts. The method is recommended for operational water flow forecasting using modern computer equipment.

1 Introduction

Ideas about the application of integration and differentiation of fractional orders have existed since the inception of the listed types of calculus. And despite the huge potential for their possible application, this area has received little attention due to the complexity of implementing numerical schemes for fractional calculus. With the development of computational algorithms, fractional calculus is beginning to be actively used, for example, in models of viscoelastic bodies, continuous media with memory, transformation of temperature and humidity in atmospheric layers, in diffusion equations and in other areas [1–5]. Its application is also possible in hydrological modeling when forecasting river runoff.

Hydrology studies a wide range of phenomena related to the distribution and movement of water in nature. Many water bodies and processes are characterized by a high degree of complexity and irregularity, making them difficult to accurately describe using traditional mathematical models. In recent decades, there has been growing practical interest in the application of fractal approaches and fractional-dimensional equations to improve the understanding, modeling, and prediction of hydrological processes.

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Much attention is paid to hydrological forecasts, since they are necessary for optimizing the economic and business activities of any country, for the construction of hydraulic structures and their operation. Forecasting is necessary to ensure the regulation of river flow, as well as to optimize the operation of hydraulic structures.

During the navigation period, short-term forecasts are regularly transmitted to shipping companies, where they are used to determine shipping levels. Timely notification of expected floods allows taking measures to protect and reduce possible adverse effects on the national economy.

It was previously concluded that the use of fractal diagnostics helps to improve the quality of short-term river runoff forecasts in individual phases of the hydrological year [6]. Knowledge of the fractal dimension helps to select the optimal prognostic equation, which contains a certain number of phase variables. The fractal dimension shows the number of these variables in the prognostic model.

The aim of the study is to improve the methodology for short-term river runoff forecasting by applying fractional differentiation taking into account the results of fractal diagnostics. To achieve the goal, the following tasks were solved:

– A database of hydrometeorological information was formed;

– Verification forecasts were issued during the spring flood and autumn floods using fractal diagnostics to select a prognostic expression in the form of a fractional differential equation;

– The obtained results were analyzed, the effectiveness of the studied method was determined and a qualitative assessment of the results was given.

2 Materials and methods

To test the approach, five hydrological posts were selected in different parts of the North-West region of the Russian Federation with different runoff formation conditions: in the eastern part of the Pinega (Пинега) River, Zasurye (Засурье) village; the Ponoy (Поной) River, Krasnoshchelye (Краснощелье) village – in the northern part, where the formation of runoff is especially influenced by the lake nature of the territory; the Tikhvinka (Тихвинка) River, Gorelukha (Горелуха) village – on the territory of the Leningrad Region; the Lovat (Ловать) River, Kholm (Холм) town – in the southern part of the North-West region; the Velikaya (Великая) River, Opochka (Опочка) town – in the southwest. Data on average daily water flow for 2019 were collected for each of these sections.

Meteorological information was also collected: average daily air temperature, daily precipitation and snow storage data for calculations during the spring flood period. The following meteorological stations were used to collect this data: Sura (Сура), Staraya Russa (Старая Русса), Krasnoshchelye (Краснощелье), Tikhvin (Тихвин) and Pushkinskiye Gory (Пушкинские Горы). The main criteria for selecting meteorological stations were data representativeness. Stations were selected as close as possible to the catchment centre of the river under study. Then, by comparing precipitation and water discharge data at each section, the parameters of the travel time and the correlation coefficient between runoff and precipitation were found to determine runoff-forming precipitation.

The following research methods were used: multivariate statistics (fractal diagnostics), models of mathematical modeling of hydrological processes (differential equations of various dimensions), methods for assessing the effectiveness of prognostic approaches.

3 Results

There are several methods for calculating fractal dimensions of time series: a method based on the correlation integral; a method based on calculating the Hurst exponent; a functional method.

The most substantiated method is the method based on the correlation integral [6]. In this method, the correlation dimension is determined:

$$D_2 = -\lim_{\varepsilon \to 0} \frac{\ln\left(\sum_{i=0}^{M(\varepsilon)} P_i^2\right)}{\ln \varepsilon} \tag{1}$$

the value P_i is the probability of a point falling into the i-th cell of size ε; P_i^2 – probability of two points falling into this cell; $\sum_{i=0}^{M(\varepsilon)} P_i^2$ – the probability that two arbitrary points lie inside a cell of size ε. This sum is the pair correlation integral:

$$C(\varepsilon) = \lim_{N \to \infty} \frac{1}{N^2} \sum_{i>j} \Theta(\varepsilon - |\vec{r}_i - \vec{r}_j|) \approx \frac{N^2(\varepsilon)}{N^2} \tag{2}$$

where $(\vec{r}_i - \vec{r}_j)$ is the distance between pairs of points ij on the trajectory of the dynamic system under consideration (here Θ = 1, if $\varepsilon - |\vec{r}_i - \vec{r}_j| \geq 0$; Θ = 0, if $\varepsilon - |\vec{r}_i - \vec{r}_j| < 0$). Thus, from (1) it follows that $C(\varepsilon) = \varepsilon^{D_2}$ or (denote $D_2 = d$):

$$d = \ln C(\varepsilon) / \ln(\varepsilon) \tag{3}$$

When using this method, the so-called pseudophase space is used. When studying a particular system, it is usually unknown how many phase variables describe it. Usually only a number of observations of some characteristic $Y(t)$ are known. And it is possible to construct the series $Y(t + \tau)$, $Y(t + 2\tau)$, ..., which, in a certain sense, allows us to obtain an analogue of the phase space.

You can create the following algorithm for calculating the fractal dimension:

1. Based on a given time series, function (2) is constructed for increasing values of the phase space dimensions n.

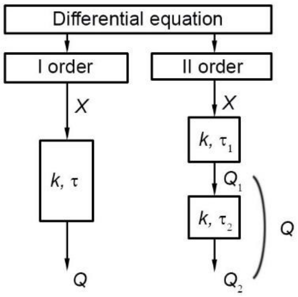
2. The "slope" d is determined by expression (3) and the dependence $d = f(n)$ is plotted.

3. The result of the calculations is a constant value of the dimension with successive shifts τ (for this purpose, so-called saturation curves are constructed). The integer immediately following the fractal dimension – the dimension of the embedding space shows the minimum number of phase variables required for reliable modeling and prediction of the processes under study. For example, it was found that $d = 0.84$. This is a characteristic of an attractor that is embedded in a one-dimensional phase space. Thus, the development of the process under consideration can be described by a model in the form of an ordinary differential equation of the first order (the embedding space is equal to unity). You can also use a fractional equation.

Fractal diagnostics helps to understand which model is more effective for issuing a forecast, taking into account the model dimension – the number of phase variables included in it. Using fractional differential equations, it is possible to improve the quality of this forecast, since there is no rounding of fractal dimensions and it is possible to take into account the participation of variables in the forecast process exactly by the amount that they contribute to the process.

Figure 1a schematically shows the components of differential equations of the first and second orders. In the figure, X – the amount of precipitation; Q_1 – surface runoff; Q_2 – groundwater runoff; Q – the total flow rate from the river catchment; τ_1 – the travel time of surface runoff; τ_2 – the travel time of groundwater runoff; τ – the total travel time; k – the coefficient taking into account losses in the catchment.

a)



b)

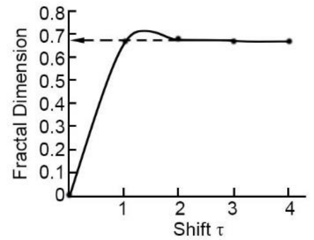


Fig. 1. Diagram of the components of differential equations (a), saturation curve showing fractal dimension (0.68) (b).

Initially, the testing of the forecasting method taking into account the fractal dimension of runoff series was carried out using dynamic models of the formation of daily water discharges in the form of differential equations of the first and second orders, respectively containing one and two phase variables and the following results were obtained:

– the accuracy of the forecasting method taking into account the fractal dimension of the series depending on the studied river for the entire period of autumn floods amounted to 35 to 70 %;

– when considering individual phases of autumn floods, it is noted that during the period of decline in river flow, the method provides a result with a accuracy of 65 to 90 %;

– the method showed a good result in forecasting runoff at the rise of the flood, the percentage of accuracy was 90 %.

Further improvement and revision of the method of applying fractal diagnostics in short-term forecasting of river runoff consisted in the use of fractal differential models, the fractional dimension of which was the result of fractal diagnostics of time series, i.e. instead of using, for example, such differentials dQ/dt, d^2Q/dt^2, $d^{0.68}Q/dt^{0.68}$ was used in accordance with the saturation curve showing the fractal dimension of the time series, see Fig. 1b.

Since fractal diagnostics showed dimensions less than one in most cases for the period of spring floods and autumn floods in the considered catchments, the predictor–corrector method was used to solve differential fractional equations [7], combining the algorithm for solving equations with fractional dimensions with the classical Adams–Bashforth–Moulton approach, which implies that the fractional differential equation is replaced by an equivalent Volterra equation [8–10]:

$$\int_{t0}^{t_{n+1}} (t_{n+1} - u)^{\beta-1} g(u) \approx \int_{t0}^{t_{n+1}} (t_{n+1} - u)^{\beta-1} g_{n+1}(u) du \,,$$

in which the integral of the right side of the equation can be written as:

$$\int_{t_0}^{t_{n+1}} (t_{n+1}-u)^{\beta-1} g(u) = \sum_{j=0}^{n+1} a_{j,n+1} g(t_j) .$$

A similar notation gives a fractional version of the one-step Adams–Moulton method (the so-called corrector formula):

$$y_{n+1} = y_0 + \frac{1}{\Gamma(\beta)} \left(\sum_{j=0}^{n} a_{j,n+1} f(t_j, y_j) + a_{n+1,n+1} f(t_{n+1}, y_{n+1}^{p}) \right) .$$

Then the predictor formula is determined:

$$y_{n+1}^{p} = y_0 + \frac{1}{\Gamma(\beta)} \left(\sum_{j=0}^{n} b_{j,n+1} f(t_j, y_j) \right) .$$

According to the presented formulas, the predictor is first calculated, then the evaluation is performed, the corrector is determined, and finally, the evaluation is performed again. Therefore, this method belongs to the PECE group (Predict, Evaluate, Correct, Evaluate).

The solution of differential equations with fractional dimension was implemented in the *MatLab* application using the author's program with the use of built-in functions (solvers).

Figure 2 shows an example of comparison of actual hydrographs and simulated ones, constructed as a result of calculations using differential models of different orders. In this example, good results are clearly visible for the model with fractional dimension.

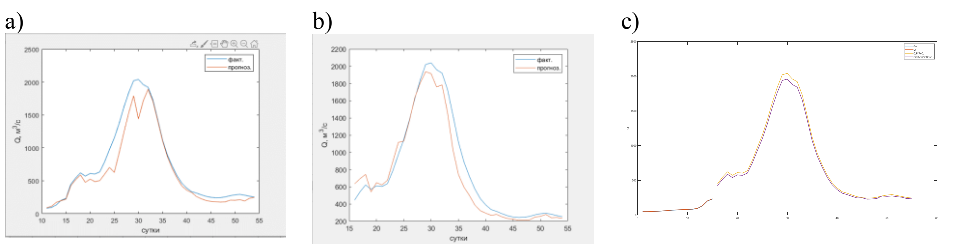


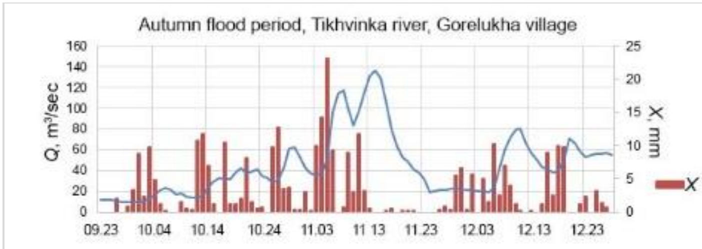
Fig. 2. Comparison of differential models of different orders in forecasting spring floods on the Pinega River, Zasurye village: a) first-order model; b) second-order model; c) model with fractional dimension (fractal dimension $n = 0.69$).

A short-term forecasting technique was tested and assessed using elements of fractal diagnostics using fractional differentials on the rivers of the Northwestern Federal District.

Fractal diagnostics was performed for a period of 15 days preceding the forecast release date, as a result of which an optimal forecasting model was selected for the lead time period. Forecasts were made with a lead time of one day.

Figures 3 and 4 show the results for some stations. Figure 4 shows satisfactory forecasts for a fractional differential equation, although it is worth noting the good results for a first-order differential equation, i.e., when rounding off the fractal dimension of the series.

a)



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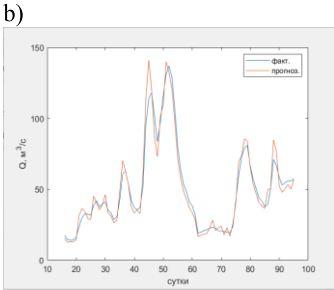

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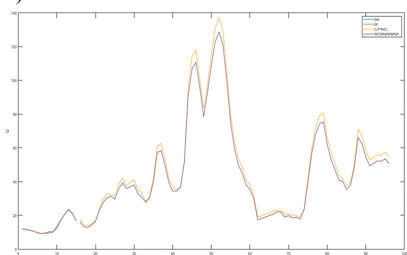
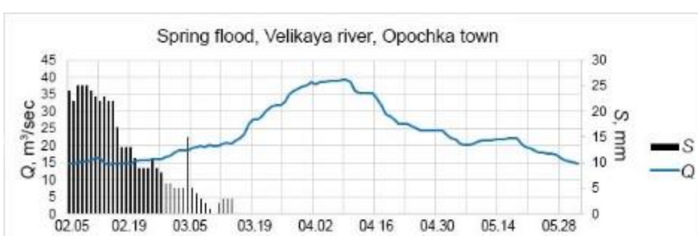


Fig. 3. Comparison of differential forecasting models of different orders for the Tikhvinka River, Gorelukha village: a) chronological graph of the autumn flood of 2019; b) first-order model; c) model with fractional dimension (fractal dimension $n = 0.94$).

Figure 4 shows the spring flood hydrograph forecast. Forecasts for models with unrounded dimensions are considered unsatisfactory forecasts, but the equation with fractional dimensions shows better results, although according to the efficiency criterion they are also considered unsatisfactory: it is clear how the model loses stability during the decline of the flood, which can be associated with errors in determining the model coefficients.

a)



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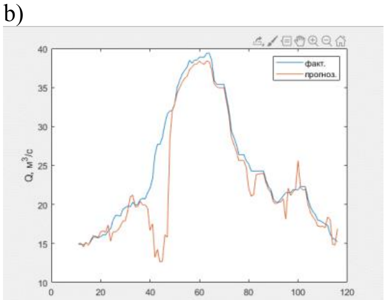

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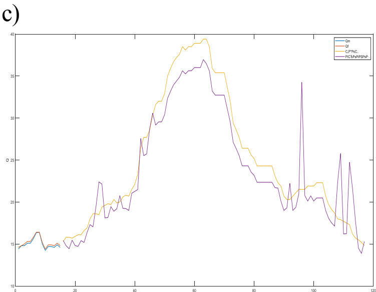

Fig. 4. Comparison of differential forecasting models of different orders for the Velikaya River, Opochka: a) chronological graph of the autumn flood of 2019; b) first-order model; c) model with fractional dimension (fractal dimension $n = 0.77$).

The results were assessed using the ratio S/σ_Δ:

$$\sigma_\Delta = \sqrt{\frac{\sum_{i=1}^{n}(\Delta_i - \overline{\Delta})^2}{n-1}}, \qquad S = \sqrt{\frac{\sum_{i=1}^{n}(Q_i - Q_i')^2}{n_f - m}},$$

where σ_Δ – standard deviation of the predicted value for the lead time period; Δ_i – change in the forecast value over the forecast lead time period; $\overline{\Delta}$ – the average value of these changes; n – the number of changes; S – the standard error of the verification forecasts; Q_i and Q_i' – the actual and predicted values, respectively; n_f – the number of forecasts; m – the number of degrees of freedom, equal to the number of constant coefficients in the prognostic equation.

Table 1 shows examples of fractal diagnostics results and an assessment of the effectiveness of forecast models in the form of first-order differential equations and fractional dimensions.

It is evident that the use of fractional differential equations shows satisfactory results in most cases and the assessment criteria are smaller compared to the results of using a first-order equation.

Table 1. Results of the evaluation of the effectiveness of forecast models in the form of differential equations

River, post (time period in 2019)	**Fractal dimension**	**S/σ_Δ**	
		I order	**fractional dimension**
r. Pinega, village Zasurye (spring flood / autumn floods)	0.69 / 0.88	1.02 / 0.80	0.53 / 0.16
r. Ponoy, village Krasnoshchelye (spring flood)	0.95	0.49	0.32
r. Tikhvinka, village Gorelukha (spring flood / autumn floods)	0.79 / 0.94	1.39 / 0.39	0.90 / 0.28
r. Lovat, town Kholm (spring flood / autumn floods)	0.82 / 0.86	0.38 / 0.74	0.36 / 0.31
r. Velikaya, town Opochka (spring flood / autumn floods)	0.77 / 0.75	4.50 / 1.30	1.55 / 1.30

The results of forecasting water discharges in the outlet sections of zonal river catchments using fractional differential equations during autumn floods and spring floods allow us to draw the following conclusions:

– an algorithm for fractal fractional differentiation applicable to short-term river runoff forecasting has been developed and tested;

– the efficiency of the method taking into account the fractal dimension in short-term forecasting of water discharges during the autumn floods has been improved by 75 %;

– the efficiency of the method taking into account the fractal dimension in short-term forecasting of water discharges during the spring floods has been improved by 100 %.

This approach is recommended for use in the presence of computer applications that allow solving differential equations of any order, including fractional ones, in real time.

4 Conclusion

The method of calculating fractal dimensions of time series is widely used in hydrology to forecast various processes, including seasonal changes in water regime. The most effective method is based on the correlation integral, which allows one to determine the fractal dimension of a dynamic attractor and minimize the number of required phase variables.

The algorithm includes the steps of constructing a correlation integral and determining the slope of the correlation dimension versus the phase space dimension. The final indicator is a stable dimension value corresponding to the minimum sufficient dimension of the embedding space. This indicator serves as the basis for selecting an appropriate forecasting model, whether integer- or fractional-order differential equations.

To improve forecast quality, it is proposed to use fractional differential equations corresponding to the fractal dimension of the time series under study. This approach avoids information loss due to dimension rounding and improves model accuracy. Using fractional differential equations allows for a better approximation of system behavior. It has been experimentally demonstrated that the use of fractal diagnostics and fractional differential equations improves forecast quality, particularly in the areas of hydrograph rise and fall during spring floods or autumn floods. Evaluation criteria confirm the superiority of fractional models over standard integer-order equations.

Thus, the proposed approach enables the effective analysis of time series of hydrological parameters and the construction of accurate forecast models using the fractal dimension of the processes under study.

Modeling hydrological processes using fractional-dimensional equations is a promising area of research, enabling improved forecast quality and the effectiveness of water resource management. Despite the implementation challenges, the development of such methods will allow hydrologists to better understand the mechanisms underlying runoff formation in river catchments.

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